

ON THE DERIVED CATEGORY OF 1-MOTIVES

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ABSTRACT. We consider the category of Deligne 1-motives over a perfect field k of exponential characteristic p and its derived category for a suitable exact structure after inverting p . As a first result, we provide a fully faithful embedding into an étale version of Voevodsky's triangulated category of geometric motives. Our second main result is that this full embedding “almost” has a left adjoint. Applying it to the motive of a variety we get a bounded complex of 1-motives, that we compute fully for smooth varieties and partly for singular varieties. Among applications, we give motivic proofs of Roitman type theorems and new cases of Deligne's conjectures on 1-motives.

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INTRODUCTION

While Grothendieck's construction of pure motives associated to smooth projective varieties over a field k is now classical [60, 29, 52, 75, 1], the construction of mixed motives associated to arbitrary k -varieties is still largely work in progress. In this direction, the first step was taken by Deligne in [25] where he defined 1-motives, which should ultimately be *mixed motives of level or dimension ≤ 1* . They form a category that we shall denote by $\mathcal{M}_1(k)$ or \mathcal{M}_1 .

Deligne's definition was motivated by Hodge theory, and he conjectured that some Hodge-theoretic constructions could be described as realisations of *a priori* constructed 1-motives [25, (10.4.1)]. In this direction, the first author and Srinivas associated in [8] homological and cohomological Albanese and Picard 1-motives $\mathrm{Alb}^-(X)$, $\mathrm{Alb}^+(X)$, $\mathrm{Pic}^-(X)$ and $\mathrm{Pic}^+(X)$ to an algebraic scheme X in characteristic zero, which extend the classical Picard and Albanese varieties. This work

was pursued in [7], where part of Deligne’s conjecture was reformulated and proven rationally (see also [72]).

A different step towards mixed motives was taken by Voevodsky who defined in [86] a *triangulated category* of motives $\mathrm{DM}_{\mathrm{gm}}^{\mathrm{eff}}(k)$. Taken with rational coefficients, this category is conjectured to have a “motivic” t -structure whose heart should be the searched-for abelian category of mixed motives.

Since $\mathcal{M}_1(k)$ is expected to be contained in such a heart, it is natural to try and relate Deligne’s and Voevodsky’s ideas. This is what Voevodsky did in [86, p. 218] (see also [84, Pretheorem 0.0.18]). Denote by $\mathcal{M}_1(k) \otimes \mathbb{Q}$ the abelian category of 1-motives up to isogeny over k . When k is perfect, Voevodsky said that there exists a *fully faithful functor*

$$D^b(\mathcal{M}_1(k) \otimes \mathbb{Q}) \hookrightarrow \mathrm{DM}_{-, \mathrm{\acute{e}t}}^{\mathrm{eff}}(k) \otimes \mathbb{Q}$$

whose essential image is the thick subcategory $d_{\leq 1} \mathrm{DM}_{\mathrm{gm}}^{\mathrm{eff}}(k) \otimes \mathbb{Q} \subseteq \mathrm{DM}_{-, \mathrm{\acute{e}t}}^{\mathrm{eff}}(k) \otimes \mathbb{Q}$ generated by motives of smooth curves.

In fact, a 1-motive may be regarded as a length 1 complex of homotopy invariant étale sheaves with transfers, so that it defines an object of Voevodsky’s triangulated category $\mathrm{DM}_{-, \mathrm{\acute{e}t}}^{\mathrm{eff}}(k)$ of étale motivic complexes [86, §3.3] to which $\mathrm{DM}_{\mathrm{gm}}^{\mathrm{eff}}(k)$ maps. This defines a functor

$$(0.1) \quad \mathcal{M}_1(k) \rightarrow \mathrm{DM}_{-, \mathrm{\acute{e}t}}^{\mathrm{eff}}(k).$$

F. Orgogozo justified Voevodsky’s assertion in [68] by extending the rational version of (0.1) to $D^b(\mathcal{M}_1(k) \otimes \mathbb{Q})$.

What started us on the present work was the desire to use the above setting to give a natural proof of Deligne’s conjecture. We achieve¹ this aim in Corollary 17.3.2 (the main result is Theorem 15.5.5, see 0.11 for details). Our method is to prove that a version of the Voevodsky-Orgogozo full embedding has a *left adjoint*, see 0.4 below. This led us on the way to a number of interesting results, for example a new light on Roïtman’s theorem (see 0.8).

Let us stress that we have to invert the exponential characteristic p of the base field k throughout. This is due to several reasons:

- Since the category $\mathrm{DM}_{-, \mathrm{\acute{e}t}}^{\mathrm{eff}}(k)$ is $\mathbb{Z}[1/p]$ -linear by [86, Prop. 3.3.3 2)], we cannot expect better comparison results.

¹Not completely, see Remark 17.3.3 for a discussion.

- To be in the spirit of Voevodsky, we want to use only the étale topology and not the fppf topology which would be more natural from the viewpoint of 1-motives. Trying to prove anything meaningful without inverting p in this context seems doomed to failure.

The basic reason why p is inverted in $\mathrm{DM}_{-, \text{ét}}^{\text{eff}}(k)$ is homotopy invariance (the Artin-Schreier exact sequence). But if one wants to deal with non homotopy invariant phenomena, Deligne 1-motives are not sufficient and one should enlarge them to include \mathbb{G}_a factors as in Laumon's 1-motives (*cf.* [9], [5]). See [11] for work in this direction.

We now give a detailed overview of the contents of this work.

0.1. The derived category of 1-motives, p -integrally. While the $\mathbb{Z}[1/p]$ -linear category $\mathcal{M}_1[1/p]$ is not an abelian category, it fully embeds into the abelian category ${}^t\mathcal{M}_1[1/p]$ of *1-motives with torsion* introduced in [7] (in characteristic 0), which makes it an exact category in the sense of Quillen (see §1.5). Its derived category $D^b(\mathcal{M}_1[1/p])$ with respect to this exact structure makes sense, and moreover the functor $D^b(\mathcal{M}_1[1/p]) \rightarrow D^b({}^t\mathcal{M}_1[1/p])$ turns out to be an equivalence (Theorem 1.6.1).

0.2. p -integral equivalence. Let $\mathrm{DM}_{\text{gm}, \text{ét}}^{\text{eff}} := \mathrm{DM}_{\text{gm}, \text{ét}}^{\text{eff}}(k)$ be the thick subcategory of $\mathrm{DM}_{-, \text{ét}}^{\text{eff}}(k)$ generated by the image of $\mathrm{DM}_{\text{gm}}^{\text{eff}}(k)$ (see Definition 2.1.1) and $d_{\leq 1} \mathrm{DM}_{\text{gm}, \text{ét}}^{\text{eff}}$ the thick subcategory of $\mathrm{DM}_{\text{gm}, \text{ét}}^{\text{eff}}$ generated by motives of smooth curves. In Theorem 2.1.2, we refine the Voevodsky-Orgogozo equivalence to an equivalence of categories

$$(0.2) \quad D^b(\mathcal{M}_1(k)[1/p]) \xrightarrow{\sim} d_{\leq 1} \mathrm{DM}_{\text{gm}, \text{ét}}^{\text{eff}}$$

0.3. Duality. Deligne's extension of Cartier duality to 1-motives [25] provides the category of 1-motives with a natural involution $M \mapsto M^*$ which extends to $D^b(\mathcal{M}_1(k)[1/p])$: see Proposition 1.8.4. This duality exchanges the category ${}^t\mathcal{M}_1[1/p]$ of §0.1 with an abelian category ${}_t\mathcal{M}_1[1/p]$ of *1-motives with cotorsion* (see §1.8).

We show in Theorem 4.5.3 that, under Tot, Deligne's Cartier duality is transformed into the involution $M \mapsto \underline{\mathrm{Hom}}(M, \mathbb{Z}(1))$ on $d_{\leq 1} \mathrm{DM}_{\text{gm}, \text{ét}}^{\text{eff}}(k)$ given by the internal (effective) Hom. Of course, this result involves biextensions.

0.4. Left adjoint. Composing (0.2) with the inclusion into $\mathrm{DM}_{\mathrm{gm},\mathrm{\acute{e}t}}^{\mathrm{eff}}(k)$, we obtain a “universal realisation functor”

$$\mathrm{Tot} : D^b(\mathcal{M}_1(k)[1/p]) \rightarrow \mathrm{DM}_{\mathrm{gm},\mathrm{\acute{e}t}}^{\mathrm{eff}}(k).$$

It was conjectured by Voevodsky ([85]; this is also implicit in [84, Preth. 0.0.18]) that, rationally, Tot has a left adjoint. We prove this in Section 6.

It is shown in Remark 5.2.2 that Tot does not have a left adjoint integrally. There is nevertheless an integral statement, which involves an interplay between the étale and the *Nisnevich* topology. Let $\alpha^* : \mathrm{DM}_{\mathrm{gm}}^{\mathrm{eff}}(k) \rightarrow \mathrm{DM}_{\mathrm{gm},\mathrm{\acute{e}t}}^{\mathrm{eff}}(k)$ be the change of topology functor. We find a functor

$$\mathrm{LAlb} : \mathrm{DM}_{\mathrm{gm}}^{\mathrm{eff}}(k) \rightarrow D^b(\mathcal{M}_1(k)[1/p])$$

together with a (natural) *motivic Albanese map*

$$(0.3) \quad a_M : \alpha^* M \rightarrow \mathrm{Tot} \mathrm{LAlb}(M)$$

for $M \in \mathrm{DM}_{\mathrm{gm}}^{\mathrm{eff}}(k)$. Thus if $(M, N) \in \mathrm{DM}_{\mathrm{gm}}^{\mathrm{eff}}(k) \times D^b(\mathcal{M}_1[1/p])$, there is a functorial homomorphism

$$(0.4) \quad \mathrm{Hom}(\mathrm{LAlb} M, N) \rightarrow \mathrm{Hom}(\alpha^* M, \mathrm{Tot}(N))$$

which is an isomorphism rationally (but not integrally in general, see Remark 5.2.2).

We give the construction of LAlb and the motivic Albanese map in Sect. 5. This is the central result of the present work.

0.5. Smooth schemes. Applying LAlb to various motives, we get interesting and intrinsically-defined 1-motives. For example, applying it to the motive $M(X)$ of a smooth scheme X , we get the *homological Albanese complex* $\mathrm{LAlb}(X) := \mathrm{LAlb}(M(X))$ of X . Its homology 1-motives $\mathrm{L}_i \mathrm{Alb}(X) := {}_t H_i(\mathrm{LAlb}(X))$ relative to the t -structure on $D^b(\mathcal{M}_1[1/p])$ with heart ${}_t \mathcal{M}_1[1/p]$ (see §0.3) are 1-motives (with cotorsion) functorially attached to X .

We then proceed to compute $\mathrm{LAlb}(X)$ for a smooth scheme X : in principle this determines LAlb on the whole of $\mathrm{DM}_{\mathrm{gm}}^{\mathrm{eff}}$, since this category is generated by the $M(X)$. It is related with the “Albanese scheme” $\mathcal{A}_{X/k}$ of [71] (extending the Serre Albanese variety of [78]) in the following way: $\mathrm{LAlb}(X)$ is a “3-extension” of $\mathcal{A}_{X/k}$ by the Cartier dual of the Néron-Severi group of X , that we define as the étale sheaf represented by cycles of codimension 1 on X modulo algebraic equivalence. (See Theorem 10.2.2.) We deduce that $\mathrm{L}_1 \mathrm{Alb}(X)$ is isomorphic to the 1-motive $\mathrm{Alb}^-(X)$ of [8] (corresponding to the Serre Albanese).

0.6. **LAlb and RPic.** Composing LAlb with duality, we obtain a contravariant functor

$$\mathrm{RPic} : \mathrm{DM}_{\mathrm{gm}}^{\mathrm{eff}}(k) \rightarrow D^b(\mathcal{M}_1(k)[1/p])$$

such that

$$\mathrm{R}^i\mathrm{Pic}(M) := {}^tH^i(\mathrm{RPic}(M)) \simeq {}_tH_i(\mathrm{LAlb}(M))^*$$

for any $M \in \mathrm{DM}_{\mathrm{gm}}^{\mathrm{eff}}(k)$. Here, ${}^tH^i$ is defined with respect to the t -structure with heart ${}^t\mathcal{M}_1[1/p]$. We call RPic the *motivic Picard functor*. We define the *cohomological Picard complex* by $\mathrm{RPic}(X) := \mathrm{RPic}(M(X))$.

0.7. **Singular schemes.** When k is of characteristic 0, the motive and motive with compact support $M(X)$ and $M^c(X)$ are defined for any k -scheme of finite type X as objects of $\mathrm{DM}_{\mathrm{gm}}^{\mathrm{eff}}(k)$, so that $\mathrm{LAlb}(X)$ and the *Borel-Moore Albanese complex* $\mathrm{LAlb}^c(X) := \mathrm{LAlb}(M^c(X))$ make sense. Still in characteristic 0 we further define, for an equidimensional scheme X of dimension n , the *cohomological Albanese complex* $\mathrm{LAlb}^*(X) := \mathrm{LAlb}(M(X)^*(n)[2n])$. We define similarly $\mathrm{RPic}^c(X) := \mathrm{RPic}(M^c(X))$ and $\mathrm{RPic}^*(X) := \mathrm{RPic}(M(X)^*(n)[2n])$. We describe some properties of these complexes in Sect. 8.

We then give some general qualitative estimates for $\mathrm{L}_i\mathrm{Alb}(X)$ in Proposition 11.3.2 (see also Proposition 13.6.1) as well as $\mathrm{L}_i\mathrm{Alb}^c(X) := {}_tH_i(\mathrm{LAlb}^c(X))$ in Proposition 11.5.2. Sect. 13 is devoted to a detailed study of $\mathrm{L}_1\mathrm{Alb}(X)$, $\mathrm{L}_1\mathrm{Alb}^c(X)$ and $\mathrm{L}_1\mathrm{Alb}^*(X)$; the main results are summarised in the introduction of this section. In particular, we prove that $\mathrm{L}_1\mathrm{Alb}(X)$ is canonically isomorphic to the 1-motive $\mathrm{Alb}^-(X)$ of [8] if X is normal or proper. Here, the interplay between LAlb and RPic (duality between Picard and Albanese) plays an essential rôle. We also prove that $\mathrm{L}_1\mathrm{Alb}^*(X) \simeq \mathrm{Alb}^+(X)$ for any X (in fact, $\mathrm{R}^1\mathrm{Pic}^*(X) \simeq \mathrm{Pic}^-(X)$ in Theorem 13.12.6).

It is quite striking that $\mathrm{L}_i\mathrm{Alb}(X)$, $\mathrm{L}_i\mathrm{Alb}^c(X)$ and $\mathrm{L}_i\mathrm{Alb}^*(X)$ are actually Deligne 1-motives for $i \leq 1$, but not in general for $i \geq 2$ (already for X smooth projective).

We also completely compute $\mathrm{L}_i\mathrm{Alb}(X)$ for any curve X , showing that $M(X)$ has a “Chow-Künneth decomposition” in $\mathrm{DM}_{\mathrm{gm},\mathrm{ét}}^{\mathrm{eff}}(k) \otimes \mathbb{Q}$ and that $\mathrm{L}_i\mathrm{Alb}(X)$ coincide with Deligne-Lichtenbaum motivic homology of the curve X (see Theorem 12.2.1, cf. [55] and [16]). Finally, we completely compute $\mathrm{L}_i\mathrm{Alb}^c(X)$ of a smooth curve X (see Theorem 12.3.1), showing that $\mathrm{L}_1\mathrm{Alb}^c(X) = H_m^1(X)(1)$ is Deligne’s motivic H^1 in [25]. Dually, we recover Deligne’s 1-motivic H^1 of any curve. With a little

more effort, one should be able to identify our computations with those of Lichtenbaum in [55] and [56].

0.8. Roïtman’s torsion theorem. If X is smooth projective, the motivic Albanese map (0.3) applied to $M = M(X)$ gives back the Albanese map from the 0-th Chow group to the rational points of the Albanese variety. This thus translates very classical mathematics to a motivic setting. When X is only smooth, we recover a generalised Albanese map from Suslin homology

$$a_X^{sing} : H_0^{sing}(X; \mathbb{Z})[1/p] \rightarrow \mathcal{A}_{X/k}(k)[1/p]$$

which was first constructed by Ramachandran [70] and Spieß-Szamuely [81].² The map a_X^{sing} is an isomorphism if $\dim(X) \leq 1$ (see Proposition 14.1.2).

We then get a very natural proof of the classical theorem of Roïtman, and even of its generalisation to open smooth varieties by Spieß-Szamuely [81, Th. 1.1] (removing their hypothesis on the existence of a smooth compactification): see Theorem 14.4.5.

We also deal with singular schemes when $\text{char } k = 0$, see Proposition 14.5.1 and its corollaries. Here there is an overlap with recent work of Geisser [37]. The works may be compared as follows: Geisser works in arbitrary characteristic and can handle p -torsion in characteristic p , but he only works with proper schemes, while the use of DM forces us to work in characteristic 0 for singular schemes, but we do handle open schemes.

Still in characteristic 0, we get a Borel-Moore version of Roïtman’s theorem as well, see Proposition 14.6.1 and its corollary.

Notably, we obtain a “cohomological” Roïtman theorem, involving torsion in a motivic cohomology group: see Corollary 14.7.4. In a recent work, Mallick [57] proves a parallel cohomological theorem, involving torsion in the Levine-Weibel Chow group (*cf.* [8, Th. 6.4.1]). Mallick works with projective schemes, but in any characteristic. Hopefully, the two theorems are compatible, see Remark 14.7.6.

0.9. The homotopy t -structure. The consideration of Voevodsky’s categories provides us with some nonobvious extra structures on the category $D^b(\mathcal{M}_1[1/p])$. Namely, the homotopy t -structure on $\text{DM}_{-, \text{ét}}^{\text{eff}}$ and the equivalence of categories (0.2) induce a *third* t -structure on $D^b(\mathcal{M}_1[1/p])$, that we also call the homotopy t -structure (Theorem

²The observation that Suslin homology is related to 1-motives is initially due to Lichtenbaum [55].

3.9.1; see also Corollary 3.9.2). Its heart is formed of so-called 1-*motivic sheaves*: their consideration is very useful for the computation of $\mathrm{LAlb}(X)$ for smooth X . This idea was pursued by Ayoub–Barbieri-Viale in [4] and by Bertapelle in [11].

0.10. Tensor structure and internal Hom. Similarly, the functor LAlb turns out to transport the tensor structure on $\mathrm{DM}_{\mathrm{gm}}^{\mathrm{eff}} \otimes \mathbb{Q}$ to a (rather degenerate) tensor structure on $D^b(\mathcal{M}_1 \otimes \mathbb{Q})$. This tensor structure is exact (for the standard t -structure), respects the weight filtration and may be computed explicitly. There is also an exact internal Hom. See Sect. 7.

0.11. A conceptual proof of Deligne’s conjectures. We now come to the main motivation of this work: a motivic understanding of Deligne’s conjectures [25, (10.4.1)].

We introduce an axiomatic framework to formulate a version of these conjectures for *any* suitable realisation functor. This involves an abstract notion of weight filtration, which is given in Appendix E. Assume given a triangulated category \mathcal{T} provided with a t -structure, with heart \mathcal{B} with a weight filtration $\mathcal{B}_{\leq n}$ (see E.7.2 for the definition) and a t -exact functor $D^b(\mathcal{B}) \rightarrow \mathcal{T}$ which is the identity on \mathcal{B} , *e.g.* $\mathcal{T} = D^b(\mathcal{B})$. Considering effective and Lefschetz objects in \mathcal{B} , we get the subcategory of level ≤ 1 objects $\mathcal{B}_{(1)} \subset \mathcal{B}^{\mathrm{eff}}$ along with a left adjoint $\mathrm{Alb}^{\mathcal{B}}$. Similarly, define $\mathcal{T}_{(1)} \subset \mathcal{T}^{\mathrm{eff}}$ (by objects with H^* in $\mathcal{B}_{(1)}$ or $\mathcal{B}^{\mathrm{eff}}$): under mild assumptions like that of a bounded t -structure, we get a left adjoint $\mathrm{LAlb}^{\mathcal{T}}$ (see 15.3.4). Suppose given a triangulated “realisation” functor $R : \mathrm{DM}_{\mathrm{gm}}^{\mathrm{eff}} \otimes \mathbb{Q} \rightarrow \mathcal{T}^{\mathrm{eff}}$ which behaves as a usual homology theory in degrees ≤ 1 , as explained in Hypotheses 15.4.1 and 15.5.2. Let $R_1 = R \mathrm{Tot}$. We then get a “base change” natural transformation

$$(0.5) \quad \mathrm{LAlb}^{\mathcal{T}} R \Rightarrow R_1 \mathrm{LAlb}^{\mathbb{Q}}$$

and, for all $i \in \mathbb{Z}$, a map

$$(0.6) \quad \mathrm{Alb}^{\mathcal{B}} H_i^R(M) \rightarrow R_1 \mathrm{L}_i \mathrm{Alb}^{\mathbb{Q}}(M)$$

where $H_i^R(M) := H_i(R(M))$, for any $M \in \mathrm{DM}_{\mathrm{gm}}^{\mathrm{eff}}(k) \otimes \mathbb{Q}$. Theorem 15.5.5 now gives an abstract version of Deligne’s conjectures: (0.5) and (0.6) are isomorphisms if, in addition, the “homological” equivalence induced by R coincides with algebraic equivalence in codimension 1 and the corresponding geometric cycle class map satisfies a Lefschetz (1,1)-type theorem (or Tate conjecture in codimension 1) for smooth projective varieties. (Note that, if (0.6) is an isomorphism for $M = M(X)$, X smooth projective, then it is an isomorphism for any M and (0.5) is an isomorphism of functors.)

0.12. Hodge structures. For X smooth over $k = \mathbb{C}$ the complex numbers, one can easily check that the 1-motive $R^i\mathrm{Pic}(X)$ has a Hodge realisation abstractly isomorphic to $H^i(X_{\mathrm{an}}, \mathbb{Z})^{\leq 1}$, the largest 1-motivic part of the mixed Hodge structure on $H^i(X_{\mathrm{an}}, \mathbb{Z})$ Tate twisted by 1 (just use 10.6.1). The above abstract framework provides such an isomorphism in a functorial way.

Namely, if $\mathcal{B} = \mathrm{MHS}$ is the category of (graded polarizable, \mathbb{Q} -linear) Deligne's mixed Hodge structures and $\mathcal{T} = D^b(\mathrm{MHS})$, the weight filtration provides \mathcal{B} with a weight structure. Then $\mathcal{B}_{(1)} = \mathrm{MHS}_{(1)}$ is the full subcategory of $\mathrm{MHS}^{\mathrm{eff}}$ given by level ≤ 1 mixed Hodge structures. Huber's [42] (covariant) Hodge realisation R , which is the one canonically defined on $\mathrm{DM}_{\mathrm{gm}}^{\mathrm{eff}}(\mathbb{C}) \otimes \mathbb{Q}$, becomes an equivalence on 1-motives by our Theorem 17.2.1. The mentioned conditions in 15.5.5 are clearly satisfied here thanks to the $(1, 1)$ -Lefschetz theorem. In the isomorphism (0.6) for $M = M(X)$, X a complex variety, the mixed Hodge structure $\mathrm{Alb}^{\mathcal{B}} H_i^R(X) = H_i(X, \mathbb{Q})_{\leq 1}$ is the largest quotient of level ≤ 1 so that (0.6) translates Deligne's conjecture on the algebraicity of this mixed Hodge structure. Similarly, $L_i\mathrm{Alb}^*(X)$ has Hodge realisation $\mathrm{Alb}^{\mathcal{B}} H_i(R(M(X)^*(n)[2n])) = H^{2n-i}(X, \mathbb{Q}(n))_{\leq 1}$, providing new cases where Deligne's conjecture holds true (up to isogeny) not included in [8], [7] or [72]. All this is Theorem 17.3.1 and its Corollary 17.3.2, see also Remark 17.3.3.

0.13. Comparing Hodge realisations. As just explained, we use Huber's Hodge realisation functor to prove Deligne's conjecture. However, before Voevodsky, Deligne defined another Hodge realisation for 1-motives in [25, 10.1.3] (see 17.4). It is a natural question whether these two realisation functors are isomorphic.

We tackle this question in Sect. 16, in an abstract setting. The main result, Theorem 16.2.2, is that given a realisation R as in 0.11 and an exact functor $\underline{T} : \mathcal{M}_1 \otimes \mathbb{Q} \rightarrow \mathcal{B}$, compatible with weights, there is at most one natural isomorphism $R_1 = R \mathrm{Tot} \Rightarrow \underline{T}$.

It would be nice to prove the existence of this isomorphism in the case of Huber's and Deligne's Hodge realisation: see Vologodsky [89] for work in this direction, as well as Remark 16.2.3. It would follow that, for any X , the 1-motives $R^i\mathrm{Pic}(X)$ are isogenous to the 1-motives $M_i(X)$ constructed in [7]. It would also follow that the 1-motive $L_1\mathrm{Alb}^*(X)$ is isogenous to $\mathrm{Alb}^+(X)$, but this already holds integrally by Theorem 13.12.6. We expect that for any separated scheme X of finite type over a field k of characteristic zero, the 1-motives $R^i\mathrm{Pic}(X)$ are *isomorphic* to the 1-motives $M_i(X)$ constructed in [7].

For X normal or proper, this also holds for $i = 1$ by the comparison results indicated in 0.7 and the results of [8].

0.14. Mixed realisations. Huber's Hodge realisation functor is only one component of her much richer mixed realisation functor [40, 42]. In Sect. 18, we show that it fits with our axiomatic approach: this yields a reasonable interpretation of the second part of Deligne's conjecture on comparison isomorphisms, see Corollaries 18.3.2 and 18.3.3.

0.15. ℓ -adic. Fortunately, in the ℓ -adic case, the issue of 0.13 turns out to be trivial. First of all, Deligne's realisation functor [25, (10.1.5)] extends to all étale sheaves without change. This provides a covariant realisation functor R^ℓ on $\mathrm{DM}_{\mathrm{ét}}^{\mathrm{eff}}(k)$, defined in §19.2. We remark that the \mathbb{Z}_ℓ -dual of R^ℓ restricts on $\mathrm{DM}_{\mathrm{gm}}^{\mathrm{eff}}(k)$ to the same functor as defined by Ivorra in [44]: we got this insight by reading [54]. This provides, in characteristic p , an ℓ -adic version of Deligne's conjectures, which depends on the Tate conjecture in codimension 1 (Theorem 19.6.1).

0.16. Going further. One could explore situations like the one around the p -adic period isomorphism. We leave these developments to the motivated reader.

0.17. Caveat. While one might hope that these results are a partial template for a future theory of mixed motives (see *e.g.* [3]), we should stress that some of them are definitely special to level ≤ 1 . Namely:

- It is succinctly pointed out in [86, §3.4 p. 215] (see [4, §2.5] for a proof) that the non finite generation of the Griffiths group prevents higher-dimensional analogues of LAlb to exist. (This goes against [84, Conj. 0.0.19].)
- Contrary to Theorem 3.9.1, the homotopy t -structure does not induce a t -structure on $d_{\leq n} \mathrm{DM}_{\mathrm{gm}, \mathrm{ét}}^{\mathrm{eff}}$ for $n \geq 2$. This can already be seen on $\mathbb{Z}(2)$, although here the homotopy sheaves are conjecturally ind-objects of $d_{\leq 2} \mathrm{DM}_{\mathrm{gm}, \mathrm{ét}}^{\mathrm{eff}}$ (see [84, §6]).

These two issues seem related in a mysterious way! However, see [4] for a possible approach to n -motivic sheaves and a conjectural picture linking the subject to the Bloch-Beilinson motivic filtration.

0.18. A small reading guide. We now offer some suggestions to the reader, hoping that they will be helpful.

One might start by quickly brushing through §1.1 to review the definition of Deligne's 1-motives, look up §1.5 to read the definition of $D^b(\mathcal{M}_1[1/p])$ and then proceed directly to Theorem 2.1.2 (full embedding), referring to Section 1 ad libitum to read the proof of this

theorem. The lengths of Sections 3 and 4 are necessary evils; they may be skipped at first reading with a look at their main results (Theorem 3.9.1, the homotopy t -structure, and Theorem 4.5.3, agreement of the two Cartier dualities).

One may then read Section 5 on the construction of LAlb and RPic (which hopefully will be pleasant enough), glance through Section 6 (their rational versions) and have a look in passing at Section 7 for the tensor structure and internal Hom on $D^b(\mathcal{M}_1 \otimes \mathbb{Q})$. After this, the reader might fly over the mostly formal sections 8 and 9, jump to Theorem 10.2.2 which computes $\mathrm{LAlb}(X)$ for a smooth scheme X , read Sections 11 and 13 on LAlb of singular schemes where he or she will have a few surprises, read Section 14 on Roïtman's theorem and its generalisations, have a well-earned rest in recovering familiar objects in Section 12 (the case of curves). Then jump to Corollary 17.3.2 which gives the Hodge realisations of $\mathrm{RPic}(X)$ and $\mathrm{LAlb}^*(X)$ for X a complex algebraic variety, look at the main results in Sect. 18 and consult Sect. 19. After which, one can backtrack to Sect. 15 and 16 to see the technical details.

And never look at the appendices.

The reader will also find an index of notations at the end.

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In all this work, k is a perfect field of exponential characteristic p . We write $\mathrm{Sm}(k)$ for the category of smooth separated schemes of finite

³As well as the Eyjafjallajökull volcano for the almost last touches to this work.

type and $Sch(k)$ for the category of all separated schemes of finite type. Since we ignore characteristic p phenomena in this paper, we invert p in the Hom groups of all categories constructed out of commutative group schemes and étale sheaves from Subsection 1.4 onwards.

Part 1. The universal realisation functor

1. THE DERIVED CATEGORY OF 1-MOTIVES

The main reference for (integral, free) 1-motives is [25, §10], see also [8, §1]. We also provide an Appendix C on 1-motives with torsion which were introduced in [7, §1]. For the derived category of 1-motives up to isogeny we refer to [86, Sect. 3.4] and [68]: here we are interested in the integral version.

1.1. Deligne 1-motives. The following terminology is handy:

1.1.1. Definition. a) An abelian sheaf L on $(Sm(k))_{\text{ét}}$ is *discrete* if it is locally constant \mathbb{Z} -constructible (*i.e.* with finitely generated geometric fibres). The full subcategory of discrete abelian sheaves on $(Sm(k))_{\text{ét}}$ is denoted by ${}^t\mathcal{M}_0(k) = {}^t\mathcal{M}_0$.

b) A *lattice* is a k -group scheme locally constant for the étale topology, with geometric fibre(s) isomorphic to a finitely generated free abelian group, *i.e.* representing a torsion-free discrete sheaf. The full subcategory of lattices is denoted by $\mathcal{M}_0(k) = \mathcal{M}_0$.

A *Deligne 1-motive* over k is a complex of group schemes

$$M = [L \xrightarrow{u} G]$$

where L is a lattice and G is a semi-abelian k -scheme. Thus G can be represented by an extension

$$0 \rightarrow T \rightarrow G \rightarrow A \rightarrow 0$$

where T is a k -torus and A is an abelian k -scheme.

As a complex, we shall place L in degree 0 and G in degree 1. Note that this convention is only partially shared by the existing literature.

A map from $M = [L \xrightarrow{u} G]$ to $M' = [L' \xrightarrow{u'} G']$ is a commutative square

$$\begin{array}{ccc} L & \xrightarrow{u} & G \\ f \downarrow & & \downarrow g \\ L' & \xrightarrow{u'} & G' \end{array}$$

in the category of group schemes. Denote by $(f, g) : M \rightarrow M'$ such a map. The natural composition of squares makes up the category of

Deligne's 1-motives. We shall denote this category by $\mathcal{M}_1(k)$. We shall usually write \mathcal{M}_1 instead of $\mathcal{M}_1(k)$, unless it is necessary to specify k . The following lemma is immediate:

1.1.2. Lemma. \mathcal{M}_1 is an idempotent complete additive category. \square

1.1.3. Definition. Let R be a commutative ring. For any additive category \mathcal{A} , we denote by $\mathcal{A} \otimes R$ the R -linear category obtained from \mathcal{A} by tensoring morphisms by R , and by $\mathcal{A} \boxtimes R$ the pseudo-abelian hull (idempotent completion) of $\mathcal{A} \otimes R$.

This distinction is useful as $\mathcal{A} \otimes R$ may not be idempotent complete even if \mathcal{A} is.

We shall also use the following category, which is technically very useful:

1.1.4. Definition. Let $\mathcal{M}_{\text{anc}}^{\text{eff}}$ denote the category given by complexes of group schemes $[L \rightarrow G]$ where L is discrete and G is a commutative algebraic group whose connected component of the identity G^0 is semi-abelian. It contains \mathcal{M}_1 as a full subcategory.

This category is studied in more detail in §C.8.

1.1.5. Proposition (cf. [68, 3.2.2]). *The inclusion $\mathcal{M}_1 \rightarrow \mathcal{M}_{\text{anc}}^{\text{eff}}$ induces an equivalence of categories*

$$e : \mathcal{M}_1 \otimes \mathbb{Q} \xrightarrow{\sim} \mathcal{M}_{\text{anc}}^{\text{eff}} \otimes \mathbb{Q}.$$

In particular, the category $\mathcal{M}_1 \otimes \mathbb{Q}$ is abelian, hence $\mathcal{M}_1 \otimes \mathbb{Q} = \mathcal{M}_1 \boxtimes \mathbb{Q}$.

Proof. (See also Lemma B.1.3.) It is enough to show that e is essentially surjective. But if $[L \rightarrow G] \in \mathcal{M}_{\text{anc}}^{\text{eff}}$, then we have a diagram

$$\begin{array}{ccc} [L^0 \rightarrow G^0] & \longrightarrow & [L_{\text{fr}}^0 \rightarrow G^0/u(L_{\text{tor}}^0)] \\ \downarrow & & \\ [L \xrightarrow{u} G] & & \end{array}$$

where the vertical (*resp.* horizontal) map is a pull-back (*resp.* a push-out) and $L_{\text{fr}}^0 := L^0/L_{\text{tor}}^0$ where L_{tor}^0 is the torsion subgroup of L^0 . Both maps are isomorphisms in $\mathcal{M}_{\text{anc}}^{\text{eff}} \otimes \mathbb{Q}$. The last assertion follows from the fact that $\mathcal{M}_{\text{anc}}^{\text{eff}} \otimes \mathbb{Q}$ is abelian (Proposition C.8.4). \square

1.1.6. Remarks. 1 (see also Def. B.1.1 c)). An *isogeny* between Deligne's 1-motives, from $M = [L \xrightarrow{u} G]$ to $M' = [L' \xrightarrow{u'} G']$ in $\mathcal{M}_1(k)$,

is a diagram of group schemes with exact columns

$$\begin{array}{ccc}
 & & 0 \\
 & & \downarrow \\
 0 & & F \\
 \downarrow & & \downarrow \\
 L & \xrightarrow{u} & G \\
 f \downarrow & & g \downarrow \\
 L' & \xrightarrow{u'} & G' \\
 \downarrow & & \downarrow \\
 E & & 0 \\
 \downarrow & & \\
 0 & &
 \end{array}$$

where F and E are finite. Isogenies become invertible in $\mathcal{M}_1 \otimes \mathbb{Q}$.

2. The category \mathcal{M}_1 of Deligne's 1-motives has kernels and cokernels (see Proposition C.1.3) but it is not abelian. This easily follows from the diagram hereabove: an isogeny has vanishing kernel and cokernel but it is not an isomorphism in \mathcal{M}_1 .

1.2. Weights and cohomological dimension. Recall that $M = [L \rightarrow G] \in \mathcal{M}_1$ has an increasing filtration by sub-1-motives as follows:

$$W_i(M) = \begin{cases} M & i \geq 0 \\ G & i = -1 \\ T & i = -2 \\ 0 & i \leq -3 \end{cases}$$

We then have $\mathrm{gr}_{-2}^W(M) = T[-1]$, $\mathrm{gr}_{-1}^W(M) = A[-1]$ and $\mathrm{gr}_0^W(M) = L$ (according to our convention of placing L in degree zero). We say that M is *pure of weight i* if $\mathrm{gr}_j^W M = 0$ for all $j \neq i$. Note that for two pure 1-motives M, M' , $\mathrm{Hom}(M, M') \neq 0$ only if they have the same weight.

1.2.1. Proposition ([68, 3.2.4]). *The category $\mathcal{M}_1 \otimes \mathbb{Q}$ is of cohomological dimension ≤ 1 , i.e. if $\mathrm{Ext}^i(M, M') \neq 0$, for $M, M' \in \mathcal{M}_1 \otimes \mathbb{Q}$, then $i = 0$ or 1 .*

Recall a sketch of the proof in [68]. (For a generalisation and a different proof, see Proposition C.12.3.) One first checks that $\mathrm{Ext}^1(M, M')$

$= 0$ if M, M' are pure of weights i, i' and $i \leq i'$. This formally reduces the issue to checking that if M, M', M'' are pure respectively of weights $0, -1, -2$, then the Yoneda product of two classes $(e_1, e_2) \in \text{Ext}^1(M, M') \times \text{Ext}^1(M', M'')$ is 0. Of course we may assume e_1 and e_2 integral. By a transfer argument, one may further reduce to k algebraically closed. Then the point is that e_1 and e_2 “glue” into a 1-motive, so are induced by a 3 step filtration on a complex of length 1; after that, it is formal to deduce that $e_2 \cdot e_1 = 0$ (cf. [SGA7, IX, Prop. 9.3.8 c)]).

1.2.2. Remark. We observe that Proposition 1.2.1 can be regarded as an algebraic version of a well-known property of $\mathcal{M}_1(\mathbb{C}) \otimes \mathbb{Q}$. Namely, $\mathcal{M}_1(\mathbb{C}) \otimes \mathbb{Q}$ can be realised as a thick abelian sub-category of \mathbb{Q} -mixed Hodge structures, see [25]. Since the latter is of cohomological dimension ≤ 1 , so is $\mathcal{M}_1(\mathbb{C}) \otimes \mathbb{Q}$ (use [58, Ch. III, Th. 9.1]).

1.3. Group schemes and sheaves with transfers.

1.3.1. Definition (cf. Def. D.1.2). We denote by $\text{HI}_{\text{ét}} = \text{HI}_{\text{ét}}(k)$ the category of homotopy invariant étale sheaves with transfers over $\text{Sm}(k)$: this is the full subcategory of the category $\text{EST}(k) = \text{Shv}_{\text{ét}}(\text{SmCor}(k))$ from [86, §3.3] consisting of those étale sheaves with transfers that are homotopy invariant.

Let G be a commutative k -group scheme. We shall denote by \underline{G} the associated étale sheaf of abelian groups. In fact, under a minor assumption, \underline{G} is an *étale sheaf with transfers*, as explained by Spieß-Szamuely [81, Proof of Lemma 3.2], cf. also Orgogozo [68, 3.1.2]. Both references use symmetric powers, hence deal only with smooth quasi-projective varieties. Here is a cheap way to extend their construction to arbitrary smooth varieties: this avoids to have to prove that $\text{DM}_{\text{gm}}^{\text{eff}}(k)$ may be presented in terms of smooth quasi-projective varieties, cf. [68, beg. of §1].

1.3.2. Lemma. *Suppose that the neutral component G^0 is quasi-projective. Then the étale sheaf \underline{G} is provided with a canonical structure of presheaf with transfers. Moreover, if G^0 is a semi-abelian variety, then \underline{G} is homotopy invariant.*

Proof. For two smooth k -varieties X, Y , we have to provide a pairing

$$c(X, Y) \otimes \underline{G}(X) \rightarrow \underline{G}(Y)$$

with the obvious compatibilities. As in [61, Ex. 2.4], it is enough to construct a good transfer $f_* : \underline{G}(W) \rightarrow \underline{G}(X)$ for any finite surjective

map $f : W \rightarrow X$ with X a normal k -variety. For X and W quasi-projective, this is done in [81] or [68]⁴. In general, cover X by affine opens U_i and let $V_i = f^{-1}(U_i)$. Since f is finite, V_i is also affine, hence transfers $\underline{G}(V_i) \rightarrow \underline{G}(U_i)$ and $\underline{G}(V_i \cap V_j) \rightarrow \underline{G}(U_i \cap U_j)$ are defined; the commutative diagram

$$\begin{array}{ccccc} 0 \rightarrow \underline{G}(W) & \longrightarrow & \prod \underline{G}(V_i) & \longrightarrow & \prod \underline{G}(V_i \cap V_j) \\ & & f_* \downarrow & & f_* \downarrow \\ 0 \rightarrow \underline{G}(X) & \longrightarrow & \prod \underline{G}(U_i) & \longrightarrow & \prod \underline{G}(U_i \cap U_j) \end{array}$$

uniquely defines the desired f_* .

The second statement of the lemma is well-known (*e.g.* [68, 3.3.1]).

□

Actually, the proof of [81, Lemma 3.2] defines a homomorphism in $\mathrm{HI}_{\mathrm{\acute{e}t}}$

$$\sigma : L_{\mathrm{\acute{e}t}}(G) \rightarrow \underline{G}$$

which is split by the obvious morphism of sheaves

$$\gamma : \underline{G} \rightarrow L_{\mathrm{\acute{e}t}}(G)$$

given by the graph of a morphism. Therefore σ is an epimorphism of sheaves. (One should be careful, however, that γ is not additive.) When \underline{G} is homotopy invariant, one deduces from it as in [81, Remark 3.3] a morphism in $\mathrm{DM}_{-, \mathrm{\acute{e}t}}^{\mathrm{eff}}(k)$

$$(1.1) \quad M_{\mathrm{\acute{e}t}}(G) = C_*(L_{\mathrm{\acute{e}t}}(G)) \rightarrow \underline{G}.$$

Let $\mathrm{HI}_{\mathrm{\acute{e}t}}^{[0,1]}$ be the category of complexes of length 1 of objects of $\mathrm{HI}_{\mathrm{\acute{e}t}}$ (concentrated in degrees 0 and 1): this is an abelian category. Lemma 1.3.2 gives us a functor

$$\begin{aligned} \mathcal{M}_{\mathrm{anc}}^{\mathrm{eff}} &\rightarrow \mathrm{HI}_{\mathrm{\acute{e}t}}^{[0,1]} \\ M &\mapsto \underline{M} \end{aligned}$$

hence, by composing with the embedding $\mathcal{M}_1 \rightarrow \mathcal{M}_{\mathrm{anc}}^{\mathrm{eff}}$ of Proposition 1.1.5, another functor

$$(1.2) \quad \mathcal{M}_1 \rightarrow \mathrm{HI}_{\mathrm{\acute{e}t}}^{[0,1]}.$$

⁴For the symmetric powers of G to exist as schemes, it suffices that G^0 be quasi-projective.

1.4. 1-motives with torsion and an exact structure on $\mathcal{M}_1[1/p]$.

Recall that, from now on, we invert the exponential characteristic p in the Hom groups of all categories constructed out of commutative k -group schemes or étale k -sheaves. This does nothing in characteristic 0.

The reader can check that most of the statements below become false if $p > 1$ is not inverted. We hope that statements integral at p may be recovered in the future by considering some kind of non-homotopy invariant motives and cohomology theories.

We start with:

1.4.1. Proposition. *Let M^\cdot be a complex of objects of $\mathcal{M}_{\text{anc}}^{\text{eff}}[1/p]$. The following conditions are equivalent:*

- (i) *The total complex $\text{Tot}(M^\cdot)$ in $C(\text{HI}_{\text{ét}})[1/p]$ (see Definition 1.3.1 and Lemma 1.3.2) is acyclic.*
- (ii) *For any $q \in \mathbb{Z}$, $H^q(M^\cdot)$ is of the form $[F^q = F^q]$, where F^q is finite.*

Proof. (ii) \Rightarrow (i) is obvious. For the converse, let $M^q = [L^q \rightarrow G^q]$ for all q . Let L^\cdot and \underline{G}^\cdot be the two corresponding “column” complexes of sheaves. Then we have a long exact sequence in $\text{HI}_{\text{ét}}$:

$$\cdots \rightarrow H^q(L^\cdot) \rightarrow H^q(\underline{G}^\cdot) \rightarrow H^q(\text{Tot}(M^\cdot)) \rightarrow H^{q+1}(L^\cdot) \rightarrow \cdots$$

The assumption implies that $H^q(L^\cdot) \xrightarrow{\sim} H^q(\underline{G}^\cdot)$ for all q . Since $H^q(L^\cdot)$ is discrete and $H^q(\underline{G}^\cdot)$ is representable by a commutative algebraic group, both must be finite. \square

We now restrict to complexes of $\mathcal{M}_1[1/p]$.

1.4.2. Definition. A complex of $\mathcal{M}_1[1/p]$ is *acyclic* if it satisfies the equivalent conditions of Proposition 1.4.1. An acyclic complex of the form $0 \rightarrow N' \xrightarrow{i} N \xrightarrow{j} N'' \rightarrow 0$ is called a *short exact sequence*.

Recall that in [7] a category of 1-motives with torsion was introduced. We shall denote it here by ${}^t\mathcal{M}_1$ in order to distinguish it from \mathcal{M}_1 . Denote by ${}^t\mathcal{M}_1^{\text{eff}}$ the effective 1-motives with torsion: ${}^t\mathcal{M}_1^{\text{eff}}$ is the full subcategory of the category $\mathcal{M}_{\text{anc}}^{\text{eff}}$ of Definition 1.1.4 consisting of the objects $[L \rightarrow G]$ where G is connected. Then ${}^t\mathcal{M}_1$ is the localisation of ${}^t\mathcal{M}_1^{\text{eff}}$ with respect to quasi-isomorphisms.

The main properties of ${}^t\mathcal{M}_1$ are recalled in Appendix C. In particular, the category ${}^t\mathcal{M}_1[1/p]$ is abelian (Theorem C.5.3) and by Proposition C.7.1 we have a full embedding

$$(1.3) \quad \mathcal{M}_1[1/p] \hookrightarrow {}^t\mathcal{M}_1[1/p]$$

which makes $\mathcal{M}_1[1/p]$ an exact subcategory of ${}^t\mathcal{M}_1[1/p]$. The following lemma is clear:

1.4.3. Lemma. *A complex $0 \rightarrow N' \xrightarrow{i} N \xrightarrow{j} N'' \rightarrow 0$ in $\mathcal{M}_1[1/p]$ is a short exact sequence in the sense of Definition 1.4.2 if and only if it is a short exact sequence for the exact structure given by (1.3).*

1.4.4. Remarks. 1) There is another, much stronger, exact structure on $\mathcal{M}_1[1/p]$, induced by its full embedding in $\mathcal{M}_{\text{anc}}^{\text{eff}}[1/p]$: it amounts to require a complex $[L \rightarrow G]$ to be exact if and only if both complexes L and G are acyclic. We shall not use this exact structure in the sequel. (See also Remark 1.8.5.)

2) Clearly, the complexes of Definition 1.4.2 do not provide $\mathcal{M}_{\text{anc}}^{\text{eff}}[1/p]$ with an exact structure. It is conceivable, however, that they define an exact structure on the localisation of $\mathcal{M}_{\text{anc}}^{\text{eff}}[1/p]$ /homotopies with respect to morphisms with acyclic kernel and cokernel.

1.5. The derived category of 1-motives.

1.5.1. Lemma. *A complex in $C(\mathcal{M}_1[1/p])$ is acyclic in the sense of Definition 1.4.2 if and only if it is acyclic with respect to the exact structure of $\mathcal{M}_1[1/p]$ provided by Lemma 1.4.3 in the sense of [10, 1.1.4] or [67, §1].*

Proof. Let $X^\cdot \in C(\mathcal{M}_1[1/p])$. Viewing X^\cdot as a complex of objects of $\mathcal{M}_{\text{anc}}^{\text{eff}}[1/p]$, we define $D^n = \text{Im}(d^n : X^n \rightarrow X^{n+1})$. Note that the D^n are Deligne 1-motives. Let $e_n : X^n \rightarrow D^n$ be the projection and $m_n : D^n \rightarrow X^{n+1}$ be the inclusion. We have half-exact sequences

$$(1.4) \quad 0 \rightarrow D^{n-1} \xrightarrow{m_{n-1}^{-1}} X^n \xrightarrow{e_n} D^n \rightarrow 0$$

with middle cohomology equal to $H^n(X^\cdot)$. Thus, if X^\cdot is acyclic in the sense of Definition 1.4.2, the sequences (1.4) are short exact which means that X^\cdot is acyclic with respect to the exact structure of $\mathcal{M}_1[1/p]$. Conversely, suppose that X^\cdot is acyclic in the latter sense. Then, by definition, we may find D'^n, e'_n, m'_n such that $d^n = m'_n e'_n$ and that the sequences analogous to (1.4) are short exact. Since $\mathcal{M}_{\text{anc}}^{\text{eff}}[1/p]$ is abelian (Proposition C.8.4), $D'^n = D^n$ and we are done. \square

From now on, we shall only say “acyclic” without further precision.

Let $K(\mathcal{M}_1[1/p])$ be the homotopy category of $C(\mathcal{M}_1[1/p])$. By [67, Lemmas 1.1 and 1.2], the full subcategory of $K(\mathcal{M}_1[1/p])$ consisting of acyclic complexes is triangulated and thick (the latter uses the fact that $\mathcal{M}_1[1/p]$ is idempotent-complete, cf. Lemma 1.1.2). Thus one may define the derived category of $\mathcal{M}_1[1/p]$ in the usual way:

1.5.2. Definition. a) The *derived category of 1-motives* is the localisation $D(\mathcal{M}_1[1/p])$ of the homotopy category $K(\mathcal{M}_1[1/p])$ with respect to the thick subcategory $A(\mathcal{M}_1[1/p])$ consisting of acyclic complexes. Similarly for $D^\pm(\mathcal{M}_1[1/p])$ and $D^b(\mathcal{M}_1[1/p])$.
b) A morphism in $C(\mathcal{M}_1[1/p])$ is a *quasi-isomorphism* if its cone is acyclic.

1.6. Torsion objects in the derived category of 1-motives. Let \mathcal{M}_0 be the category of lattices (see Definition 1.1.1): the inclusion functor $\mathcal{M}_0[1/p] \xrightarrow{A} \mathcal{M}_1[1/p]$ provides it with the structure of an exact subcategory of $\mathcal{M}_1[1/p]$. Moreover, the embedding

$$\mathcal{M}_0[1/p] \xrightarrow{B} {}^t\mathcal{M}_0[1/p]$$

is clearly exact, where ${}^t\mathcal{M}_0$ is the abelian category of discrete étale sheaves (see Definition 1.1.1 again). In fact, we also have an exact functor

$$\begin{aligned} {}^t\mathcal{M}_0[1/p] &\xrightarrow{C} {}^t\mathcal{M}_1[1/p] \\ L &\mapsto [L \rightarrow 0]. \end{aligned}$$

Hence an induced diagram of triangulated categories:

$$\begin{array}{ccc} D^b(\mathcal{M}_0[1/p]) & \xrightarrow{B} & D^b({}^t\mathcal{M}_0[1/p]) \\ A \downarrow & & c \downarrow \\ D^b(\mathcal{M}_1[1/p]) & \xrightarrow{D} & D^b({}^t\mathcal{M}_1[1/p]). \end{array}$$

1.6.1. Theorem. *In the above diagram*

- a) *B and D are equivalence of categories.*
- b) *A and C are fully faithful; restricted to torsion objects they are equivalences of categories.*

(For the notion of torsion objects, see Proposition B.2.1.)

Proof. a) For B , this follows from Proposition A.1.2 provided we check that any object M in ${}^t\mathcal{M}_0[1/p]$ has a finite left resolution by objects in $\mathcal{M}_0[1/p]$. In fact M has a length 1 resolution: let E/k be a finite Galois extension of group Γ such that the Galois action on M factors through Γ . Since M is finitely generated, it is a quotient of some power of $\mathbb{Z}[\Gamma]$, and the kernel is a lattice. Exactly the same argument works for D .

b) By a) it is sufficient to prove that C is fully faithful. It suffices to verify that the criterion of Proposition A.1.4 is verified by the full embedding ${}^t\mathcal{M}_0[1/p] \rightarrow {}^t\mathcal{M}_1[1/p]$.

Let $[L \rightarrow 0] \hookrightarrow [L' \rightarrow G']$ be a monomorphism in ${}^t\mathcal{M}_1[1/p]$. We may assume that it is given by an effective map. The assumption implies that $L \rightarrow L'$ is mono: it then suffices to compose with the projection $[L' \rightarrow G'] \rightarrow [L' \rightarrow 0]$.

It remains to show that A is essentially surjective on torsion objects. Let $X = [C^\bullet \rightarrow G^\bullet] \in D^b(\mathcal{M}_1[1/p])$, and let $n > 0$ be such that $n1_X = 0$. Arguing as in the proof of Proposition 1.4.1, this implies that the cohomology sheaves of both C^\bullet and G^\bullet are killed by some possibly larger integer m . We have an exact triangle

$$[0 \rightarrow G^\bullet] \rightarrow X \rightarrow [C^\bullet \rightarrow 0] \xrightarrow{+1}$$

which leaves us to show that $[0 \rightarrow G^\bullet]$ is in the essential image of C . Let q be the smallest integer such that $G^q \neq 0$: we have an exact triangle

$$\{G^q \rightarrow \operatorname{Im} d^q\} \rightarrow G^\bullet \rightarrow \{0 \rightarrow G^{q+1}/\operatorname{Im} d^q \rightarrow \dots\} \xrightarrow{+1}$$

(here we use curly braces in order to avoid confusion with the square braces used for 1-motives). By descending induction on q , the right term is in the essential image, hence we are reduced to the case where G^\bullet is of length 1. Then $d^q : G^q \rightarrow G^{q+1}$ is epi and $\mu := \operatorname{Ker} d^q$ is finite and locally constant⁵. Consider the diagram in $K^b(\mathcal{M}_{\text{anc}}^{\text{eff}}[1/p])$

$$\left[\begin{array}{c} 0 \\ \downarrow \\ 0 \end{array} \rightarrow \begin{array}{c} G^q \\ \downarrow \\ G^{q+1} \end{array} \right] \leftarrow \left[\begin{array}{c} 0 \\ \downarrow \\ \mu \end{array} \rightarrow \begin{array}{c} G^q \\ \parallel \\ G^q \end{array} \right] \leftarrow \left[\begin{array}{c} L_1 \\ \downarrow \\ L_0 \end{array} \rightarrow \begin{array}{c} G^q \\ \parallel \\ G^q \end{array} \right] \rightarrow \left[\begin{array}{c} L_1 \\ \downarrow \\ L_0 \end{array} \rightarrow \begin{array}{c} 0 \\ \downarrow \\ 0 \end{array} \right]$$

where $L_1 \rightarrow L_0$ is a resolution of μ by lattices (see proof of a)). Clearly all three maps are quasi-isomorphisms, which implies that the left object is quasi-isomorphic to the right one on $D^b(\mathcal{M}_1[1/p])$. \square

1.6.2. Corollary. *Let A be a subring of \mathbb{Q} containing $1/p$. Then the natural functor*

$$D^b(\mathcal{M}_1[1/p]) \otimes A \rightarrow D^b(\mathcal{M}_1 \otimes A)$$

is an equivalence of categories. These categories are idempotent-complete for any A .

Proof. By Proposition B.4.1, this is true by replacing the category $\mathcal{M}_1[1/p]$ by ${}^t\mathcal{M}_1[1/p]$. On the other hand, the same argument as above shows that the functor $D^b(\mathcal{M}_1 \otimes A) \rightarrow D^b({}^t\mathcal{M}_1 \otimes A)$ is an equivalence. This shows the first statement; the second one follows from the fact that D^b of an abelian category is idempotent-complete. \square

⁵Note that this is true even if m is divisible by the characteristic of k , since we only consider sheaves over *smooth* k -schemes.

1.7. Discrete sheaves and permutation modules. The following proposition will be used in §2.6.a.

1.7.1. Proposition. *Let G be a profinite group. Denote by $D_c^b(G)$ the derived category of finitely generated (topological discrete) G -modules. Then $D_c^b(G)$ is thickly generated by \mathbb{Z} -free permutation modules.*

Proof. The statement says that the smallest thick subcategory \mathcal{T} of $D_c^b(G)$ which contains permutation modules is equal to $D_c^b(G)$. Let M be a finitely generated G -module: to prove that $M \in \mathcal{T}$, we immediately reduce to the case where G is finite. Let $\overline{M} = M/M_{\text{tors}}$. Realise $\overline{M} \otimes \mathbb{Q}$ as a direct summand of $\mathbb{Q}[G]^n$ for n large enough. Up to scaling, we may assume that the image of \overline{M} in $\mathbb{Q}[G]^n$ is contained in $\mathbb{Z}[G]^n$ and that there exists a submodule N of $\mathbb{Z}[G]^n$ such that $\overline{M} \cap N = 0$ and $\overline{M} \oplus N$ is of finite index in $\mathbb{Z}[G]^n$. This reduces us to the case where M is *finite*. Moreover, we may assume that M is ℓ -primary for some prime ℓ .

Let S be a Sylow ℓ -subgroup of G . Recall that there exist two inverse isomorphisms

$$\begin{aligned} \varphi : \mathbb{Z}[G] \otimes_{\mathbb{Z}[S]} M &\xrightarrow{\sim} \text{Hom}_{\mathbb{Z}[S]}(\mathbb{Z}[G], M) \\ \varphi(g \otimes m)(\gamma) &= \begin{cases} \gamma g m & \text{if } \gamma g \in S \\ 0 & \text{if } \gamma g \notin S. \end{cases} \\ \psi : \text{Hom}_{\mathbb{Z}[S]}(\mathbb{Z}[G], M) &\xrightarrow{\sim} \mathbb{Z}[G] \otimes_{\mathbb{Z}[S]} M \\ \psi(f) &= \sum_{g \in S \backslash G} g^{-1} \otimes f(g). \end{aligned}$$

On the other hand, we have the obvious unit and counit homomorphisms

$$\begin{aligned} \eta : M &\rightarrow \text{Hom}_{\mathbb{Z}[S]}(\mathbb{Z}[G], M) \\ \eta(m)(g) &= gm \\ \varepsilon : \mathbb{Z}[G] \otimes_{\mathbb{Z}[S]} M &\rightarrow M \\ \varepsilon(g \otimes m) &= gm. \end{aligned}$$

It is immediate that

$$\varepsilon \circ \psi \circ \eta = (G : S).$$

Since $(G : S)$ is prime to ℓ , this shows that M is a direct summand of the induced module $\mathbb{Z}[G] \otimes_{\mathbb{Z}[S]} M \simeq \text{Hom}_{\mathbb{Z}[S]}(\mathbb{Z}[G], M)$. But it is well-known (see *e.g.* [80, §8.3, cor. to Prop. 26]) that M , as an S -module, is a successive extension of trivial S -modules. Any trivial

torsion S -module has a length 1 resolution by trivial torsion-free S -modules. Since the “induced module” functor is exact, this concludes the proof. \square

1.8. Cartier duality and 1-motives with cotorsion. We now introduce a new category ${}_t\mathcal{M}_1$:

1.8.1. Definition. We denote by ${}_t\mathcal{M}_1^{\text{eff}}$ the full subcategory of $\mathcal{M}_{\text{anc}}^{\text{eff}}$ consisting of those $[L \rightarrow G]$ such that L is a lattice and G is an extension of an abelian variety by a group of multiplicative type, and by ${}_t\mathcal{M}_1$ its localisation with respect to quasi-isomorphisms. An object of ${}_t\mathcal{M}_1$ is called a *1-motive with cotorsion*.

Recall that Deligne [25, §10.2.11-13] (cf. [8, 1.5]) defined a self-duality on the category \mathcal{M}_1 , that he called *Cartier duality*. The following facts elucidate the introduction of the category ${}_t\mathcal{M}_1$.

1.8.2. Lemma. *Let Γ be a group of multiplicative type, L its Cartier dual and A an abelian variety (over $k = \bar{k}$). We have an isomorphism*

$$\tau : \text{Ext}(A, \Gamma) \xrightarrow{\simeq} \text{Hom}(L, \text{Pic}^0(A))$$

given by the canonical “pushout” mapping.

Proof. Displaying L as an extension of L_{fr} by L_{tor} denote the corresponding torus by $T := \text{Hom}(L_{\text{fr}}, \mathbb{G}_m)$ and let $F := \text{Hom}(L_{\text{tor}}, \mathbb{G}_m)$ be the dual finite group. We obtain a map of short exact sequences

$$\begin{array}{ccccccc} 0 \rightarrow \text{Ext}(A, T) & \rightarrow & \text{Ext}(A, \Gamma) & \rightarrow & \text{Ext}(A, F) \rightarrow 0 \\ \tau_{\text{fr}} \downarrow & & \tau \downarrow & & \tau_{\text{tor}} \downarrow \\ 0 \rightarrow \text{Hom}(L_{\text{fr}}, \text{Pic}^0(A)) & \rightarrow & \text{Hom}(L, \text{Pic}^0(A)) & \rightarrow & \text{Hom}(L_{\text{tor}}, \text{Pic}^0(A)) \rightarrow 0. \end{array}$$

Now τ_{fr} is an isomorphism by the classical Weil-Barsotti formula, *i.e.* $\text{Ext}(A, \mathbb{G}_m) \cong \text{Pic}^0(A)$, and τ_{tor} is an isomorphism since the Néron-Severi group of A is free: $\text{Hom}(L_{\text{tor}}, \text{Pic}^0(A)) = \text{Hom}(L_{\text{tor}}, \text{Pic}(A)) = H^1(A, F) = \text{Ext}(A, F)$ (cf. [62, 4.20]). \square

1.8.3. Lemma. *Cartier duality on \mathcal{M}_1 extends to a contravariant additive functor*

$$(\)^* : {}_t\mathcal{M}_1^{\text{eff}} \rightarrow {}_t\mathcal{M}_1^{\text{eff}}$$

which sends a q.i. to a q.i.

Proof. The key point is that $\text{Ext}(-, \mathbb{G}_m)$ vanishes on discrete sheaves (cf. [62, 4.17]), hence Cartier duality extends to an exact duality between discrete sheaves and groups of multiplicative type.

To define the functor, we proceed as usual (see [8, 1.5]): starting with $M = [L \xrightarrow{u} A] \in {}_t\mathcal{M}_1^{\text{eff}}$, let G^u be the extension of the dual abelian

variety A^* by the Cartier dual L^* of L given by Lemma 1.8.2 (note that G^u may be described as the group scheme which represents the functor associated to $\text{Ext}(M, \mathbb{G}_m)$). We define $M^* = [0 \rightarrow G^u] \in {}_t\mathcal{M}_1^{\text{eff}}$. For a general $M = [L \xrightarrow{u} G] \in {}_t\mathcal{M}_1^{\text{eff}}$, with G an extension of A by T , the extension M of $[L \xrightarrow{\bar{u}} A]$ by the toric part $[0 \rightarrow T]$ provides the corresponding extension $G^{\bar{u}}$ of A' by L^* and a boundary map

$$u^* : \text{Hom}(T, \mathbb{G}_m) \rightarrow \text{Ext}([L \xrightarrow{\bar{u}} A], \mathbb{G}_m) = G^{\bar{u}}(k)$$

which defines $M^* \in {}_t\mathcal{M}_1^{\text{eff}}$.

For a quasi-isomorphism $M \rightarrowtail M'$ with kernel $[F \xrightarrow{\bar{\cdot}} F]$ for a finite group F , cf. (C.2), the quotient $[L \xrightarrow{\bar{u}} A] \rightarrowtail [L' \xrightarrow{\bar{u}'} A']$ has kernel $[F \rightarrowtail F_A]$ where $F_A := \text{Ker}(A \rightarrowtail A')$ and the following is a pushout

$$\begin{array}{ccccccc} 0 \rightarrow \text{Hom}(T', \mathbb{G}_m) & \rightarrow & \text{Hom}(T, \mathbb{G}_m) & \rightarrow & \text{Hom}(F_T, \mathbb{G}_m) & \rightarrow & 0 \\ (u')^* \downarrow & & u^* \downarrow & & \parallel \downarrow & & \\ 0 \rightarrow \text{Ext}([L' \xrightarrow{\bar{u}'} A'], \mathbb{G}_m) & \rightarrow & \text{Ext}([L \xrightarrow{\bar{u}} A], \mathbb{G}_m) & \rightarrow & \text{Ext}([F \rightarrowtail F_A], \mathbb{G}_m) & \rightarrow & 0 \end{array}$$

where $F_T := \text{Ker}(T \rightarrowtail T')$. \square

1.8.4. Proposition. *a) The functor of Lemma 1.8.3 induces an anti-equivalence of categories*

$$(\)^* : {}_t\mathcal{M}_1[1/p] \xrightarrow{\sim} {}_t\mathcal{M}_1[1/p].$$

b) The category ${}_t\mathcal{M}_1[1/p]$ is abelian and the two functors of a) are exact.

c) Cartier duality on $\mathcal{M}_1[1/p]$ is an exact functor, hence induces a triangulated self-duality on $D^b(\mathcal{M}_1[1/p])$.

Proof. a) The said functor exists by Lemma 1.8.3, and it is clearly additive. Let us prove that it is i) essentially surjective, ii) faithful and iii) full.

i) We proceed exactly as in the proof of Lemma 1.8.3, taking an $[L' \rightarrow G'] \in {}_t\mathcal{M}_1[1/p]$, and writing G' explicitly as an extension of an abelian variety by a group of multiplicative type.

ii) We reduce to show that the functor of Lemma 1.8.3 is faithful by using that Lemma C.2.3 is also true in ${}_t\mathcal{M}_1^{\text{eff}}[1/p]$ (dual proof). By additivity, we need to prove that if $f : M_0 \rightarrow M_1$ is mapped to 0, then $f = 0$. But, by construction, f^* sends the multiplicative type part of M_1^* to that of M_0^* .

iii) Let $M_0 = [L_0 \rightarrow G_0]$, $M_1 = [L_1 \rightarrow G_1]$ in ${}^t\mathcal{M}_1^{\text{eff}}[1/p]$, and let $f : M_1^* \rightarrow M_0^*$ be (for a start) an effective map. We have a diagram

$$\begin{array}{ccccccc} 0 & \longrightarrow & \Gamma_1 & \longrightarrow & G'_1 & \longrightarrow & A'_1 \longrightarrow 0 \\ & & & & \downarrow f_G & & \\ 0 & \longrightarrow & \Gamma_0 & \longrightarrow & G'_0 & \longrightarrow & A'_0 \longrightarrow 0 \end{array}$$

where $M_i^* = [L'_i \rightarrow G'_i]$, A'_i is the dual of the abelian part of M_i and Γ_i is the dual of L_i . If f_G maps Γ_1 to Γ_2 , there is no difficulty to get an (effective) map $g : M_0 \rightarrow M_1$ such that $g^* = f$. In general we reduce to this case: let μ be the image of $f_G(\Gamma_1)$ in A'_0 : this is a finite group. Let now $A'_2 = A'_0/\mu$, so that we have a commutative diagram

$$\begin{array}{ccccccc} 0 & \longrightarrow & \Gamma_0 & \longrightarrow & G'_0 & \longrightarrow & A'_0 \longrightarrow 0 \\ & & \downarrow & & \parallel \downarrow & & \downarrow \\ 0 & \longrightarrow & \Gamma_2 & \longrightarrow & G'_0 & \longrightarrow & A'_2 \longrightarrow 0 \end{array}$$

where $\mu = \text{Ker}(A'_0 \rightarrow A'_2) = \text{Coker}(\Gamma_0 \rightarrow \Gamma_2)$. By construction, f_G induces maps $f_\Gamma : \Gamma_1 \rightarrow \Gamma_2$ and $f_A : A'_1 \rightarrow A'_2$.

Consider the object $M_2 = [L_2 \rightarrow G_2] \in {}^t\mathcal{M}_1^{\text{eff}}[1/p]$ obtained from (L'_0, Γ_2, A'_2) and the other data by the same procedure as in the proof of Lemma 1.8.3. We then have a q.i. $s : M_2 \rightarrow M_0$ with kernel $[\mu = \mu]$ and a map $g : M_2 \rightarrow M_1$ induced by (f_L, f_Γ, f_A) , and $(gs^{-1})^* = f$.

If f is a q.i., clearly g is a q.i.; this concludes the proof of fullness.

b) Since ${}^t\mathcal{M}_1[1/p]$ is abelian, ${}^t\mathcal{M}_1[1/p]$ is abelian by a). Equivalences of abelian categories are automatically exact.

c) One checks as for ${}^t\mathcal{M}_1[1/p]$ that the inclusion of $\mathcal{M}_1[1/p]$ into ${}^t\mathcal{M}_1[1/p]$ induces the exact structure of $\mathcal{M}_1[1/p]$. Then, thanks to b), Cartier duality preserves exact sequences of $\mathcal{M}_1[1/p]$, which means that it is exact on $\mathcal{M}_1[1/p]$. \square

1.8.5. Remarks. 1) Cartier duality does not preserve the strong exact structure of Remark 1.4.4 1). For example, let A be an abelian variety, $a \in A(k)$ a point of order $m > 1$ and $B = A/\langle a \rangle$. Then the sequence

$$0 \rightarrow [\mathbb{Z} \rightarrow 0] \xrightarrow{m} [\mathbb{Z} \xrightarrow{f} A] \rightarrow [0 \rightarrow B] \rightarrow 0,$$

with $f(1) = a$, is exact in the sense of Definition 1.4.2 but not in the sense of Remark 1.4.4. However, its dual

$$0 \rightarrow [0 \rightarrow B^*] \rightarrow [0 \rightarrow G] \rightarrow [0 \rightarrow \mathbb{G}_m] \rightarrow 0$$

is exact in the strong sense. Taking the Cartier dual of the latter sequence, we come back to the former.

2) One way to better understand what happens in Lemma 1.8.3 and Proposition 1.8.4 would be to introduce a category $\widetilde{{}_t\mathcal{M}_1^{\text{eff}}}$, whose objects are quintuples (L, u, G, A, Γ) with L a lattice, Γ a group of multiplicative type, A an abelian variety, G an extension of A by Γ and u a morphism from L to G . Morphisms in $\widetilde{{}_t\mathcal{M}_1^{\text{eff}}}$ are additive and respect all these structures. There is an obvious functor $(L, u, G, A, \Gamma) \mapsto [L \xrightarrow{u} G]$ from $\widetilde{{}_t\mathcal{M}_1^{\text{eff}}}$ to ${}_t\mathcal{M}_1^{\text{eff}}$, the functor of Lemma 1.8.3 lifts to an anti-isomorphism of categories ${}_t\mathcal{M}_1^{\text{eff}} \xrightarrow{\sim} \widetilde{{}_t\mathcal{M}_1^{\text{eff}}}$ and the localisation of $\widetilde{{}_t\mathcal{M}_1^{\text{eff}}}[1/p]$ with respect to the images of q.i. of ${}_t\mathcal{M}_1^{\text{eff}}[1/p]$ is equivalent to ${}_t\mathcal{M}_1[1/p]$. We leave details to the interested reader.

Dually to Theorem 1.6.1, we now have:

1.8.6. Theorem. *The natural functor $\mathcal{M}_1[1/p] \rightarrow {}_t\mathcal{M}_1[1/p]$ is fully faithful and induces an equivalence of categories*

$$D^b(\mathcal{M}_1[1/p]) \xrightarrow{\sim} D^b({}_t\mathcal{M}_1[1/p]).$$

Moreover, Cartier duality exchanges ${}_t\mathcal{M}_1[1/p]$ and ${}_t\mathcal{M}_1[1/p]$ inside the derived category $D^b(\mathcal{M}_1[1/p])$.

Proof. This follows from Theorem 1.6.1 and Proposition 1.8.4. \square

1.8.7. Notation. For $C \in D^b(\mathcal{M}_1[1/p])$, we write ${}^tH^n(C)$ (*resp.* ${}_tH^n(C)$) for its cohomology objects relative to the t -structure with heart ${}_t\mathcal{M}_1[1/p]$ (*resp.* ${}_t\mathcal{M}_1[1/p]$). We also write tH_n for ${}^tH^{-n}$ and ${}_tH_n$ for ${}_tH^{-n}$.

Thus we have *two* t -structures on $D^b(\mathcal{M}_1[1/p])$ which are exchanged by Cartier duality; naturally, these two t -structures coincide after tensoring with \mathbb{Q} . In Section 3, we shall introduce a third t -structure, of a completely different kind: see Corollary 3.9.2.

We shall also come back to Cartier duality in Section 4.

1.9. How not to invert p . This has been done by Alessandra Bertapelle [11]. She defines a larger variant of ${}_t\mathcal{M}_1$ by allowing finite connected k -group schemes in the component of degree 0. Computing in the fppf topology, she checks that the arguments provided in Appendix C carry over in this context and yield in particular an integral analogue to Theorem C.5.3. Also, the analogue of (1.3) is fully faithful integrally, hence an exact structure on \mathcal{M}_1 ; she also checks that the analogue of Theorem 1.6.1 holds integrally.

In particular, her work provides an exact structure on \mathcal{M}_1 , hence an integral definition of $D^b(\mathcal{M}_1)$. One could check that this exact structure can be described *a priori* using Proposition 1.4.1 and Lemma 1.4.3, and working with the fppf topology.

It is likely that the duality results of §1.8 also extend to Bertapelle's context.

2. UNIVERSAL REALISATION

2.1. Statement of the theorem. The derived category of 1-motives up to isogeny can be realised in Voevodsky's triangulated category of motives. With rational coefficients, this is part of Voevodsky's Pretheorem 0.0.18 in [84] and claimed in [86, Sect. 3.4, on page 218]. Details of this fact appear in Orgogozo [68]. In this section we shall give a p -integral version of this theorem, where p is the exponential characteristic of k , using the étale version of Voevodsky's category. By Lemma 1.3.2, any 1-motive $M = [L \rightarrow G]$ may be regarded as a complex of homotopy invariant étale sheaves with transfers. By Lemma D.1.3, $M[1/p] := M \otimes_{\mathbb{Z}} \mathbb{Z}[1/p]$ is a complex of strictly homotopy invariant étale sheaves with transfers; this defines a functor

$$(2.1) \quad \begin{aligned} \mathcal{M}_1(k) &\rightarrow \mathrm{DM}_{-, \text{ét}}^{\text{eff}}(k) \\ M &\mapsto M[1/p]. \end{aligned}$$

(see [86, Sect. 3] for motivic complexes).

From now on, we will usually drop the mention of k from the notation for the various categories of motives encountered.

2.1.1. Definition. We denote by $\mathrm{DM}_{\text{gm}, \text{ét}}^{\text{eff}}$ the thick subcategory of $\mathrm{DM}_{-, \text{ét}}^{\text{eff}}$ generated by the image of $\mathrm{DM}_{\text{gm}}^{\text{eff}}$ under the “change of topology” functor

$$\alpha^* : \mathrm{DM}_{-}^{\text{eff}} \rightarrow \mathrm{DM}_{-, \text{ét}}^{\text{eff}}$$

of [86, §3.3]. We set $M_{\text{ét}}(X) := \alpha^* M(X)$.

2.1.2. Theorem. *Let p be the exponential characteristic of k . The functor (2.1) extends to a fully faithful triangulated functor*

$$T : D^b(\mathcal{M}_1[1/p]) \rightarrow \mathrm{DM}_{-, \text{ét}}^{\text{eff}}$$

where the left hand side was defined in §1.4. Its essential image is the thick subcategory $d_{\leq 1} \mathrm{DM}_{\text{gm}, \text{ét}}^{\text{eff}}$ of $\mathrm{DM}_{\text{gm}, \text{ét}}^{\text{eff}}$ generated by motives of smooth curves.

The proof is in several steps.

2.2. Construction of T . We follow Orgogozo. Clearly, the embedding (1.2) extends to a functor

$$C^b(\mathcal{M}_1) \rightarrow C^b(\mathrm{HI}_{\text{ét}}^{[0,1]}).$$

By Lemma A.2.1, we have a canonical functor $C^b(\mathrm{HI}_{\mathrm{\acute{e}t}}^{[0,1]}) \xrightarrow{\mathrm{Tot}} D^b(\mathrm{HI}_{\mathrm{\acute{e}t}})$, and there is a canonical composite functor

$$D^b(\mathrm{HI}_{\mathrm{\acute{e}t}}) \xrightarrow{\otimes_{\mathbb{Z}} \mathbb{Z}[1/p]} D^b(\mathrm{HI}_{\mathrm{\acute{e}t}}^s) \rightarrow \mathrm{DM}_{-, \mathrm{\acute{e}t}}^{\mathrm{eff}}$$

where $\mathrm{HI}_{\mathrm{\acute{e}t}}^s$ is the category of strictly homotopy invariant étale sheaves with transfers (see Def. D.1.2 and Proposition D.1.4). To get T , we are therefore left to prove

2.2.1. Lemma. *The composite functor*

$$C^b(\mathcal{M}_1) \rightarrow C^b(\mathrm{HI}_{\mathrm{\acute{e}t}}^{[0,1]}) \xrightarrow{\mathrm{Tot}} D^b(\mathrm{HI}_{\mathrm{\acute{e}t}})$$

factors through $D^b(\mathcal{M}_1[1/p])$.

Proof. It is a general fact that a homotopy in $C^b(\mathcal{M}_1)$ is mapped to a homotopy in $C^b(\mathrm{HI}_{\mathrm{\acute{e}t}}^{[0,1]})$, and therefore goes to 0 in $D^b(\mathrm{HI}_{\mathrm{\acute{e}t}})$, so that the functor already factors through $K^b(\mathcal{M}_1)$. The lemma now follows from Lemma 1.5.1. \square

2.3. Full faithfulness. It is sufficient by Proposition B.2.4 to show that $T \otimes \mathbb{Q}$ and T_{tors} are fully faithful.

For the first fact, we reduce to [68, 3.3.3 ff]. We have to be a little careful since Orgogozo's functor is not quite the same as our functor: Orgogozo sends C to $\mathrm{Tot}(C)$ while we send it to $\mathrm{Tot}(C)[1/p]$, but the map $\mathrm{Tot}(C) \rightarrow \mathrm{Tot}(C)[1/p]$ is an isomorphism in $\mathrm{DM}_{-, \mathrm{\acute{e}t}}^{\mathrm{eff}} \otimes \mathbb{Q}$ as we shall see in Proposition 6.1.1 (see also Remark 2.7.2 2)).

For the reader's convenience we sketch the proof of [68, 3.3.3 ff]: it first uses the equivalence of categories

$$\mathrm{DM}_{-}^{\mathrm{eff}} \otimes \mathbb{Q} \xrightarrow{\sim} \mathrm{DM}_{-, \mathrm{\acute{e}t}}^{\mathrm{eff}} \otimes \mathbb{Q}$$

of [86, Prop. 3.3.2] (*cf.* Proposition 6.1.1). One then reduces to show that the morphisms

$$\mathrm{Ext}^i(M, M') \rightarrow \mathrm{Hom}(\mathrm{Tot}(M), \mathrm{Tot}(M')[i])$$

are isomorphisms for any pure 1-motives M, M' and any $i \in \mathbb{Z}$. This is done by a case-by-case inspection, using the fact [86, 3.1.9 and 3.1.12] that in $\mathrm{DM}_{-}^{\mathrm{eff}} \otimes \mathbb{Q}$

$$\mathrm{Hom}(M(X), C) \otimes \mathbb{Q} = \mathbb{H}_{\mathrm{Zar}}^0(X, C) \otimes \mathbb{Q}$$

for any smooth variety X . The key points are that 1) for such X we have $H_{\mathrm{Zar}}^i(X, \mathbb{G}_m) = 0$ for $i > 1$ and for an abelian variety A , $H_{\mathrm{Zar}}^i(X, A) = 0$ for $i > 0$ because the sheaf A is flasque, and 2) that any abelian variety is up to isogeny a direct summand of the Jacobian

of a curve. This point will also be used for the essential surjectivity below.

For the second fact, the argument in the proof of [86, Prop. 3.3.3 1] shows that the functor $\mathrm{DM}_{-, \text{ét}}^{\text{eff}} \rightarrow D^-(\mathrm{Shv}((\mathrm{Spec} k)_{\text{ét}}))$ which takes a complex of sheaves on $\mathrm{Sm}(k)_{\text{ét}}$ to its restriction to $(\mathrm{Spec} k)_{\text{ét}}$ is an equivalence of categories on the full subcategories of objects of prime-to- p torsion. The conclusion then follows from Theorem 1.6.1.

2.4. Gersten's principle. We want to formalise here an important computational method which goes back to Gersten's conjecture but was put in a wider perspective and systematic use by Voevodsky. For the étale topology it replaces advantageously (but not completely) the recourse to proper base change.

2.4.1. Proposition. *a) Let C be a complex of presheaves with transfers on $\mathrm{Sm}(k)$ with homotopy invariant cohomology presheaves. Suppose that $C(K) := \varinjlim_{k(U)=K} C(U)$ is acyclic for any function field K/k . Then the associated complex of Zariski sheaves C_{Zar} is acyclic.*

b) Let $f : C \rightarrow D$ be a morphism of complex of presheaves with transfers on $\mathrm{Sm}(k)$ with homotopy invariant cohomology presheaves. Suppose that for any function field K/k , $f(K) : C(K) \rightarrow D(K)$ is a quasi-isomorphism. Then $f_{\mathrm{Zar}} : C_{\mathrm{Zar}} \rightarrow D_{\mathrm{Zar}}$ is a quasi-isomorphism.

c) The conclusions of a) and b) hold for the étale topology if their hypotheses are weakened by replacing K by K_s , a separable closure of K .

Proof. a) Let $F = H^q(C)$ for some $q \in \mathbb{Z}$, and let X be a smooth k -variety with function field K . By [88, Cor. 4.18], $F(\mathcal{O}_{X,x}) \hookrightarrow F(K)$ for any $x \in X$, hence $F_{\mathrm{Zar}} = 0$. b) follows from a) by considering the cone of f . c) is seen similarly. \square

2.5. An important computation. Recall that the category $\mathrm{DM}_{-, \text{ét}}^{\text{eff}}$ is provided with a partial internal Hom denoted by $\underline{\mathrm{Hom}}_{\text{ét}}$, defined on pairs (M, M') with $M \in \mathrm{DM}_{\mathrm{gm}, \text{ét}}^{\text{eff}}$: it is defined analogously to the one of [86, Prop. 3.2.8] for the Nisnevich topology. We need:

2.5.1. Definition. Let $X \in \mathrm{Sch}(k)$. We denote by $\pi_0(X)$ the largest étale k -scheme such that the structural map $X \rightarrow \mathrm{Spec} k$ factors through $\pi_0(X)$.

(The existence of $\pi_0(X)$ is obvious, for example by Galois descent: see [30, Ch. I, §4.6].)

2.5.2. Proposition. *Let $f : C \rightarrow \operatorname{Spec} k$ be a smooth projective k -curve. Then, in $\operatorname{DM}_{-, \text{ét}}^{\text{eff}}$:*

a) *There is a canonical isomorphism*

$$\underline{\operatorname{Hom}}_{\text{ét}}(M_{\text{ét}}(C), \mathbb{Z}(1)[2]) \simeq R_{\text{ét}} f_* \mathbb{G}_m[1/p][1].$$

b) *we have*

$$R_{\text{ét}}^q f_* \mathbb{G}_m[1/p] = \begin{cases} R_{\pi_0(C)/k} \mathbb{G}_m[1/p] & \text{for } q = 0 \\ \underline{\operatorname{Pic}}_{C/k}[1/p] & \text{for } q = 1 \\ 0 & \text{else.} \end{cases}$$

Here, $R_{\pi_0(C)/k}$ denotes the Weil restriction of scalars from $\pi_0(C)$ to k .

c) *The morphism*

$$M_{\text{ét}}(C) \rightarrow \underline{\operatorname{Hom}}_{\text{ét}}(M_{\text{ét}}(C), \mathbb{Z}(1)[2])$$

induced by the class $\Delta_C \in \operatorname{Hom}(M_{\text{ét}}(C) \otimes M_{\text{ét}}(C), \mathbb{Z}(1)[2])$ of the diagonal is an isomorphism.

This is [68, Cor. 3.1.6] with three differences: 1) the fppf topology should be replaced by the étale topology; p must be inverted (cf. Corollary D.1.6); 3) the truncation is not necessary since C is a curve.

Proof. a) is the étale analogue of [86, Prop. 3.2.8] since $\mathbb{Z}_{\text{ét}}(1) = \mathbb{G}_m[1/p][-1]$ (see Corollary D.1.6) and $f^*(\mathbb{G}_{m,k}) = \mathbb{G}_{m,C}$ for the big étale sites. In b), the isomorphisms for $q = 0, 1$ are clear; for $q > 2$, we reduce by Gersten's principle (Prop. 2.4.1) to stalks at separably closed fields, and then the result is classical [SGA4, IX (4.5)].

It remains to prove c). Recall that its Nisnevich analogue is true in $\operatorname{DM}_{\text{gm}}^{\text{eff}}$ ([86, Th. 4.3.2 and Cor. 4.3.6], but see [43, App. B] to avoid resolution of singularities). Let $\alpha^* : \operatorname{DM}_{-}^{\text{eff}} \rightarrow \operatorname{DM}_{-, \text{ét}}^{\text{eff}}$ be the change of topology functor (cf. [61, Remark 14.3]). By b), the natural morphism

$$(2.2) \quad \alpha^* \underline{\operatorname{Hom}}_{\text{Nis}}(M(C), \mathbb{Z}(1)) \rightarrow \underline{\operatorname{Hom}}_{\text{ét}}(\alpha^* M(C), \mathbb{Z}(1))$$

is an isomorphism. Hence the result. \square

2.6. Essential image. We proceed in two steps:

2.6.a. *The essential image of T is contained in $\mathcal{T} := d_{\leq 1} \operatorname{DM}_{\text{gm}, \text{ét}}^{\text{eff}}$.* It is sufficient to prove that $T(N) \in \mathcal{T}$ for N a 1-motive of type $[L \rightarrow 0]$, $[0 \rightarrow G]$ (G a torus) or $[0 \rightarrow A]$ (A an abelian variety). For the first type, this follows from Proposition 1.7.1. For the second type, Proposition 1.7.1 applied to the character group of G shows that $T([0 \rightarrow G])$ is contained in the thick subcategory generated by permutation tori, which is clearly contained in \mathcal{T} .

It remains to deal with the third type. If $A = J(C)$ for a smooth projective curve C having a rational k -point c , then $T([0 \rightarrow A]) = A[-1]$ is the direct summand of $M(C)[-1]$ (determined by c) corresponding to the pure motive $h^1(C)$, so belongs to \mathcal{T} . If $A \rightarrow A'$ is an isogeny, then Proposition 1.7.1 implies that $A[-1] \in \mathcal{T} \iff A'[-1] \in \mathcal{T}$. In general we may write A as the quotient of a jacobian $J(C)$. Let B be the connected part of the kernel: by complete reducibility there exists a third abelian variety $B' \subseteq J(C)$ such that $B + B' = J(C)$ and $B \cap B'$ is finite. Hence $B \oplus B' \in \mathcal{T}$, $B' \in \mathcal{T}$ and finally $A \in \mathcal{T}$ since it is isogenous to B' .

2.6.b. *The essential image of T contains \mathcal{T} .* It suffices to show that $M(X)$ is in the essential image of T if X is smooth projective irreducible of dimension 0 or 1. Let E be the field of constants of X . If $X = \text{Spec } E$, $M(X)$ is the image of $[R_{E/k}\mathbb{Z} \rightarrow 0]$. If X is a curve, we apply Proposition 2.5.2: by c) it suffices to show that the sheaves of b) are in the essential image of T . We have already observed that $R_{E/k}\mathbb{G}_m[1/p]$ is in the essential image of T . We then have a short exact sequence

$$0 \rightarrow R_{E/k}J(X)[1/p] \rightarrow \underline{\text{Pic}}_{X/k}[1/p] \rightarrow R_{E/k}\mathbb{Z}[1/p] \rightarrow 0.$$

Both the kernel and the cokernel in this extension belong to the image of T , and the proof is complete. \square

2.7. The universal realisation functor.

2.7.1. **Definition.** Define the *universal realisation functor*

$$\text{Tot} : D^b(\mathcal{M}_1[1/p]) \rightarrow \text{DM}_{\text{gm},\text{ét}}^{\text{eff}}$$

to be the composition of the equivalence of categories of Theorem 2.1.2 and the embedding $d_{\leq 1} \text{DM}_{\text{gm},\text{ét}}^{\text{eff}} \rightarrow \text{DM}_{\text{gm},\text{ét}}^{\text{eff}}$.

2.7.2. **Remarks.** 1) In view of Theorem 1.8.6, the equivalence of Theorem 2.1.2, yields *two* “motivic” t -structures on $d_{\leq 1} \text{DM}_{\text{gm},\text{ét}}^{\text{eff}}$: one with heart ${}^t\mathcal{M}_1[1/p]$ and the other with heart ${}_t\mathcal{M}_1[1/p]$. We shall describe a third one, the homotopy t -structure, in Theorem 3.9.1.

2) In what follows we shall frequently commit an abuse of notation in writing \underline{G} rather than $\underline{G}[1/p]$, etc. for the image of (say) a semi-abelian variety in $\text{DM}_{\text{gm},\text{ét}}^{\text{eff}}$ by the functor Tot . This is to keep notation light. A more mathematical justification is that, according to Proposition D.1.5, the functor T is naturally isomorphic to the composition

$$\begin{aligned} D^b(\mathcal{M}_1[1/p]) &\rightarrow D^b(\text{HI}_{\text{ét}}^{[0,1]}[1/p]) \rightarrow D^b(\text{HI}_{\text{ét}}[1/p]) \\ &\rightarrow D^-(\text{Shv}_{\text{ét}}(\text{Sm}(k))[1/p]) \xrightarrow{C_*} \text{DM}_{-, \text{ét}}^{\text{eff}} \end{aligned}$$

which (apparently) does not invert p on objects.

3. 1-MOTIVIC SHEAVES AND THE HOMOTOPY t -STRUCTURE

We recall the blanket assumption that p is inverted in all Hom groups.

3.1. Some useful lemmas. Except for Proposition 3.1.7, this subsection is in the spirit of [77, Ch. VII].

Let G be a commutative k -group scheme, and let us write \underline{G} for the associated sheaf of abelian groups for a so far unspecified Grothendieck topology. Let also \mathcal{F} be another sheaf of abelian groups. We then have:

- $\mathrm{Ext}^1(\underline{G}, \mathcal{F})$ (an Ext of sheaves);
- $H^1(G, \mathcal{F})$ (cohomology of the scheme G);
- $\overline{H}^2(G, \mathcal{F})$: this is the homology of the complex

$$\mathcal{F}(G) \xrightarrow{d^1} \mathcal{F}(G \times G) \xrightarrow{d^2} \mathcal{F}(G \times G \times G)$$

where the differentials are the usual ones.

3.1.1. Proposition. *There is an exact sequence (defining A)*

$$0 \rightarrow A \rightarrow \mathrm{Ext}^1(\underline{G}, \mathcal{F}) \xrightarrow{b} H^1(G, \mathcal{F}) \xrightarrow{c} H^1(G \times G, \mathcal{F})$$

and an injection

$$0 \rightarrow A \xrightarrow{a} \overline{H}^2(G, \mathcal{F}).$$

Proof. Let us first define the maps a, b, c :

- c is given by $p_1^* + p_2^* - \mu^*$, where μ is the group law of G .
- For b : let \mathcal{E} be an extension of \underline{G} by \mathcal{F} . We have an exact sequence

$$\mathcal{E}(G) \rightarrow \underline{G}(G) \rightarrow H^1(G, \mathcal{F}).$$

Then $b([\mathcal{E}])$ is the image of 1_G by the connecting homomorphism. Alternatively, we may think of \mathcal{E} as an \mathcal{F} -torsor over G by forgetting its group structure.

- For a : we have $b([\mathcal{E}]) = 0$ if and only if 1_G has an antecedent $s \in \mathcal{E}(G)$. By Yoneda, this s determines a section $s : \underline{G} \rightarrow \mathcal{E}$ of the projection. The defect of s to be a homomorphism gives a well-defined element of $\overline{H}^2(G, \mathcal{F})$ by the usual cocycle computation: this is $a([\mathcal{E}])$.

Exactness is checked by inspection. \square

3.1.2. Remark. It is not clear whether a is surjective.

3.1.3. Proposition. *Suppose that the map*

$$\mathcal{F}(G) \oplus \mathcal{F}(G) \xrightarrow{(p_1^*, p_2^*)} \mathcal{F}(G \times G)$$

is surjective. Then $\overline{H}^2(G, \mathcal{F}) = 0$.

Proof. Let $\gamma \in \mathcal{F}(G \times G)$ be a 2-cocycle. We may write $\gamma = p_1^* \alpha + p_2^* \beta$. The cocycle condition implies that α and β are constant. Hence γ is constant, and it is therefore a 2-coboundary (of itself). \square

3.1.4. Example. \mathcal{F} locally constant, G smooth, the topology = the étale topology. Then the condition of Proposition 3.1.3 is verified. We thus get an isomorphism

$$\mathrm{Ext}^1(\underline{G}, \mathcal{F}) \xrightarrow{\sim} H_{\mathrm{\acute{e}t}}^1(G, \mathcal{F})_{\mathrm{mult}}$$

with the group of multiplicative classes in $H_{\mathrm{\acute{e}t}}^1(G, \mathcal{F})$.

3.1.5. Lemma. *Let G be a semi-abelian k -variety and L a locally constant \mathbb{Z} -constructible étale sheaf with torsion-free geometric fibres. Then $\mathrm{Ext}^1(\underline{G}, L) = 0$.*

Proof. By the Ext spectral sequence, it suffices to show that $\mathcal{H}om(\underline{G}, L) = \mathcal{E}xt(\underline{G}, L) = 0$. This reduces us to the case $L = \mathbb{Z}$. Then the first vanishing is obvious and the second follows from Example 3.1.4. \square

3.1.6. Lemma. *Let $\mathcal{E} \in \mathrm{Ext}^1(\underline{G}, \mathbb{G}_m)$ and let $g \in G(k)$. Denote by τ_g the left translation by g . Then $\tau_g^* b(\mathcal{E}) = b(\mathcal{E})$. Here b is the map of Proposition 3.1.1.*

Proof. By Hilbert's theorem 90, g lifts to an $e \in \mathcal{E}(k)$. Then τ_e induces a morphism from the \mathbb{G}_m -torsor $b(\mathcal{E})$ to the \mathbb{G}_m -torsor $\tau_g^* b(\mathcal{E})$: this morphism must be an isomorphism. \square

For the proof of Theorem 3.3.1 below we shall need the case $i = 2$ of the following proposition, which unfortunately cannot be proven with the above elementary methods.

3.1.7. Proposition. *Let G be a smooth commutative algebraic k -group and L a discrete k -group scheme. Let $\mathcal{A} = \mathrm{Shv}_{\mathrm{\acute{e}t}}(\mathrm{Sm}(k))$ be the category of abelian étale sheaves on the category of smooth k -varieties. Then, for any $i \geq 2$, the group $\mathrm{Ext}_{\mathcal{A}}^i(\underline{G}, L)$ is torsion.*

Proof. Considering the connected part G^0 of G , we reduce to the case where G is connected, hence geometrically connected. We now turn to the techniques of [21]⁶: using essentially the Eilenberg-Mac Lane spectrum associated to \underline{G} , Breen gets two spectral sequences $'E_r^{p,q}$ and $''E_r^{p,q}$ converging to the same abutment, with

- $''E_2^{p,1} = \mathrm{Ext}_{\mathcal{A}}^p(\underline{G}, L)$;
- $''E_2^{p,q}$ is torsion for $q \neq 1$;
- $'E_2^{p,q}$ is the p -th cohomology group of a complex involving terms of the form $H_{\mathrm{\acute{e}t}}^q(G^a, L)$.

⁶We thank L. Illusie for pointing out this reference.

(In [21], Breen works with the fppf topology but his methods carry over here without any change: see remark in *loc. cit.* top of p. 34.) It follows from [31, (2.1)] that $H_{\text{ét}}^q(G^a, L)$ is torsion for any $q > 0$: to see this easily, reduce to the case where L is constant by a transfer argument involving a finite extension of k . Hence $'E_2^{p,q}$ is torsion for $q > 0$. On the other hand, since G is geometrically connected, so are its powers G^a , which implies that $H^0(G^a, L) = H^0(k, L)$ for any a . Since the complex giving $'E_2^{*,0}$ is just the bar complex, we get that $'E_2^{0,0} = L(k)$ and $'E_2^{p,0} = 0$ for $p > 0$. Thus all degree > 0 terms of the abutment are torsion, and the conclusion follows. \square

3.2. 1-motivic sheaves.

3.2.1. Definition. An étale sheaf \mathcal{F} on $Sm(k)$ is *1-motivic* if there is a morphism of sheaves

$$(3.1) \quad \underline{G} \xrightarrow{b} \mathcal{F}$$

where G is a semi-abelian variety and $\text{Ker } b, \text{Coker } b$ are discrete (see Definition 1.1.1).

We denote by Shv_0 the full subcategory of $\text{Shv}_{\text{ét}}(Sm(k))[1/p]$ consisting of discrete sheaves and by Shv_1 the full subcategory of $\text{Shv}_{\text{ét}}(Sm(k))[1/p]$ consisting of 1-motivic sheaves.

3.2.2. Remark. The category Shv_0 is equivalent to the category ${}^t\mathcal{M}_0[1/p]$ of Definition 1.1.1.

3.2.3. Proposition. *a) In Definition 3.2.1 we may choose b such that $\text{Ker } b$ is torsion-free: we then say that b is normalised.*

b) Given two 1-motivic sheaves $\mathcal{F}_1, \mathcal{F}_2$, normalised morphisms $b_i : \underline{G}_i \rightarrow \mathcal{F}_i$ and a map $\varphi : \mathcal{F}_1 \rightarrow \mathcal{F}_2$, there exists a unique homomorphism of group schemes $\varphi_G : G_1 \rightarrow G_2$ such that the diagram

$$\begin{array}{ccc} \underline{G}_1 & \xrightarrow{b_1} & \mathcal{F}_1 \\ \varphi_G \downarrow & & \downarrow \varphi \\ \underline{G}_2 & \xrightarrow{b_2} & \mathcal{F}_2 \end{array}$$

commutes.

c) Given a 1-motivic sheaf \mathcal{F} , a pair (G, b) with b normalised is uniquely determined by \mathcal{F} .

d) The categories Shv_0 and Shv_1 are exact abelian subcategories of $\text{Shv}_{\text{ét}}(Sm(k))$.

Proof. a) If $\text{Ker } b$ is not torsion-free, simply divide G by the image of its torsion.

b) We want to construct a commutative diagram

$$(3.2) \quad \begin{array}{ccccccccc} 0 & \longrightarrow & L_1 & \xrightarrow{a_1} & \underline{G}_1 & \xrightarrow{b_1} & \mathcal{F}_1 & \xrightarrow{c_1} & E_1 & \longrightarrow & 0 \\ & & \varphi_L \downarrow & & \varphi_G \downarrow & & \varphi \downarrow & & \varphi_E \downarrow & & \\ 0 & \longrightarrow & L_2 & \xrightarrow{a_2} & \underline{G}_2 & \xrightarrow{b_2} & \mathcal{F}_2 & \xrightarrow{c_2} & E_2 & \longrightarrow & 0 \end{array}$$

where $L_i = \text{Ker } b_i$ and $E_i = \text{Coker } b_i$. It is clear that $c_2 \varphi b_1 = 0$: this proves the existence of φ_E . We also get a homomorphism of sheaves $\underline{G}_1 \rightarrow \underline{G}_2/L_2$, which lifts to $\varphi_G : \underline{G}_1 \rightarrow \underline{G}_2$ by Lemma 3.1.5, hence φ_L .

From the construction, it is clear that φ_E is uniquely determined by φ and that φ_L is uniquely determined by φ_G . It remains to see that φ_G is unique. Let φ'_G be another choice. Then $b_2(\varphi_G - \varphi'_G) = 0$, hence $(\varphi_G - \varphi'_G)(\underline{G}_1) \subseteq L_2$, which implies that $\varphi_G = \varphi'_G$.

c) Follows from b).

d) The case of Shv_0 is obvious. For Shv_1 , given a map φ as in b), we want to show that $\mathcal{F}_3 = \text{Ker } \varphi$ and $\mathcal{F}_4 = \text{Coker } \varphi$ are 1-motivic. Let $G_3 = (\text{Ker } \varphi_G)^0$ and $G_4 = \text{Coker } \varphi_G$: we get induced maps $b_i : \underline{G}_i \rightarrow \mathcal{F}_i$ for $i = 3, 4$. An easy diagram chase shows that $\text{Ker } b_i$ and $\text{Coker } b_i$ are both discrete. \square

Here is an extension of Proposition 3.2.3 which elucidates the structure of Shv_1 somewhat:

3.2.4. Theorem. *a) Let SAb be the category of semi-abelian k -varieties. Then the inclusion functor*

$$\begin{aligned} \text{SAb} &\rightarrow \text{Shv}_1 \\ G &\mapsto \underline{G} \end{aligned}$$

has a right adjoint/left inverse γ ; the counit of this adjunction is given by (3.1) (with b normalised). The functor γ is faithful and “exact up to isogenies”. For a morphism $\varphi \in \text{Shv}_1$, $\gamma(\varphi) = \varphi_G$ is an isogeny if and only if $\text{Ker } \varphi$ and $\text{Coker } \varphi \in \text{Shv}_0$. In particular, γ induces an equivalence of categories

$$\text{Shv}_1 / \text{Shv}_0 \xrightarrow{\sim} \text{SAb} \otimes \mathbb{Q}$$

where $\text{SAb} \otimes \mathbb{Q}$ is the category of semi-abelian varieties up to isogenies.

b) The inclusion functor $\text{Shv}_0 \rightarrow \text{Shv}_1$ has a left adjoint/left inverse π_0 ; the unit of this adjunction is given by $\text{Coker } b$ in (3.1). The right exact functor

$$(\pi_0)_{\mathbb{Q}} : \text{Shv}_1 \rightarrow \text{Shv}_0 \otimes \mathbb{Q}$$

has one left derived functor $(\pi_1)_{\mathbb{Q}}$ given by $\text{Ker } b$ in (3.1).

Proof. a) The only delicate thing is the exactness of γ up to isogenies. This means that, given a short exact sequence $0 \rightarrow \mathcal{F}' \rightarrow \mathcal{F} \rightarrow \mathcal{F}'' \rightarrow 0$ of 1-motivic sheaves, the sequence

$$0 \rightarrow \gamma(\mathcal{F}') \rightarrow \gamma(\mathcal{F}) \rightarrow \gamma(\mathcal{F}'') \rightarrow 0$$

is half exact and the middle homology is finite. This follows from a chase in the diagram

$$\begin{array}{ccccccccc} 0 & \longrightarrow & L' & \xrightarrow{a'} & \underline{G}' & \xrightarrow{b'} & \mathcal{F}' & \xrightarrow{c'} & E' & \longrightarrow & 0 \\ & & \downarrow & & \downarrow & & \downarrow & & \downarrow & & \\ 0 & \longrightarrow & L & \xrightarrow{a} & \underline{G} & \xrightarrow{b} & \mathcal{F} & \xrightarrow{c'} & E & \longrightarrow & 0 \\ & & \downarrow & & \downarrow & & \downarrow & & \downarrow & & \\ 0 & \longrightarrow & L'' & \xrightarrow{a''} & \underline{G}'' & \xrightarrow{b''} & \mathcal{F}'' & \xrightarrow{c''} & E'' & \longrightarrow & 0 \end{array}$$

of which we summarize the main points: (1) $G' \rightarrow G$ is injective because its kernel is the same as $\text{Ker}(L' \rightarrow L)$. (2) $G \rightarrow G''$ is surjective because (i) $\text{Hom}(\underline{G}'' \rightarrow \text{Coker}(E' \rightarrow E)) = 0$ and (ii) if $L'' \rightarrow \text{Coker}(\underline{G} \rightarrow \underline{G}'')$ is onto, then this cokernel is 0. (3) The middle homology is finite because the image of $\text{Ker}(\underline{G}' \rightarrow \underline{G}) \rightarrow E'$ must be finite.

In b), the existence and characterisation of $(\pi_1)_{\mathbb{Q}}$ follows from the exactness of γ in a). \square

3.2.5. Remark. One easily sees that π_1 does not exist integrally. Rather, it exists as a functor to the category of pro-objects of Shv_0 . (Actually to a finer subcategory: compare [79].)

3.3. Extensions of 1-motivic sheaves. The aim of this subsection is to prove:

3.3.1. Theorem. *The categories Shv_0 and Shv_1 are stable by extensions in the abelian category $\text{Shv}_{\text{ét}}(\text{Sm}(k))[1/p]$.*

Proof. For simplicity, let us write $\mathcal{A} := \text{Shv}_{\text{ét}}(\text{Sm}(k))[1/p]$ as in Proposition 3.1.7. The statement is obvious for Shv_0 . Let us now show that Shv_1 is closed under extensions in \mathcal{A} . Let $\mathcal{F}_1, \mathcal{F}_2$ be as in (3.2) (no map given between them). We have to show that the injection

$$(3.3) \quad \text{Ext}_{\text{Shv}_1}^1(\mathcal{F}_2, \mathcal{F}_1) \hookrightarrow \text{Ext}_{\mathcal{A}}^1(\mathcal{F}_2, \mathcal{F}_1)$$

is surjective. This is certainly so in the following special cases:

- (1) \mathcal{F}_1 and \mathcal{F}_2 are semi-abelian varieties;
- (2) \mathcal{F}_2 is semi-abelian and \mathcal{F}_1 is discrete (see Example 3.1.4).

For $m > 1$, consider

$$\mathcal{F}^m = \text{Coker}(L_1 \xrightarrow{(a_1, m)} \underline{G}_1 \oplus L_1)$$

so that we have two exact sequences

$$\begin{aligned} 0 &\longrightarrow \underline{G}_1 \xrightarrow{(1_{G_1}, 0)} \mathcal{F}^m \longrightarrow L_1/m \longrightarrow 0 \\ 0 &\longrightarrow L_1 \xrightarrow{(a_1, 0)} \mathcal{F}^m \longrightarrow \underline{G}_1/L_1 \oplus L_1/m \longrightarrow 0. \end{aligned}$$

The first one shows that (3.3) is surjective for $(\mathcal{F}_2, \mathcal{F}_1) = (\underline{G}_2, \mathcal{F}^m)$. Let us now consider the commutative diagram with exact rows associated to the second one, for an unspecified m :

$$(3.4) \quad \begin{array}{ccccc} \text{Ext}_{\text{Shv}_1}^1(\underline{G}_2, \mathcal{F}^m) & \longrightarrow & \text{Ext}_{\text{Shv}_1}^1(\underline{G}_2, \underline{G}_1/L_1 \oplus L_1/m) & \longrightarrow & \text{Ext}_{\text{Shv}_1}^2(\underline{G}_2, L_1) \\ \wr \downarrow & & \downarrow & & \downarrow \\ \text{Ext}_{\mathcal{A}}^1(\underline{G}_2, \mathcal{F}^m) & \longrightarrow & \text{Ext}_{\mathcal{A}}^1(\underline{G}_2, \underline{G}_1/L_1 \oplus L_1/m) & \xrightarrow{\delta^m} & \text{Ext}_{\mathcal{A}}^2(\underline{G}_2, L_1). \end{array}$$

Note that the composition

$$\text{Ext}_{\mathcal{A}}^1(\underline{G}_2, \underline{G}_1/L_1) \longrightarrow \text{Ext}_{\mathcal{A}}^1(\underline{G}_2, \underline{G}_1/L_1 \oplus L_1/m) \xrightarrow{\delta^m} \text{Ext}_{\mathcal{A}}^2(\underline{G}_2, L_1)$$

coincides with the boundary map δ associated to the exact sequence

$$0 \rightarrow L_1 \rightarrow \underline{G}_1 \rightarrow \underline{G}_1/L_1 \rightarrow 0.$$

Let $e \in \text{Ext}_{\mathcal{A}}^1(\underline{G}_2, \underline{G}_1/L_1)$. By Proposition 3.1.7, $f = \delta(e)$ is torsion. Choose now m such that $mf = 0$. Then there exists $e' \in \text{Ext}_{\mathcal{A}}^1(\underline{G}_2, L_1/m)$ which bounds to f via the Ext exact sequence associated to the exact sequence of sheaves

$$0 \rightarrow L_1 \xrightarrow{m} L_1 \rightarrow L_1/m \rightarrow 0.$$

Since $\delta^m(e, -e') = 0$, (3.4) shows that $(e, -e')$ comes from the left, which shows that (3.3) is surjective for $(\mathcal{F}_2, \mathcal{F}_1) = (\underline{G}_2, \underline{G}_1/L_1)$.

By Lemma 3.1.5, in the commutative diagram

$$\begin{array}{ccc} \text{Ext}_{\text{Shv}_1}^1(\underline{G}_2, \underline{G}_1/L_1) & \longrightarrow & \text{Ext}_{\text{Shv}_1}^1(\underline{G}_2, \mathcal{F}_1) \\ \wr \downarrow & & \downarrow \\ \text{Ext}_{\mathcal{A}}^1(\underline{G}_2, \underline{G}_1/L_1) & \longrightarrow & \text{Ext}_{\mathcal{A}}^1(\underline{G}_2, \mathcal{F}_1) \end{array}$$

the horizontal maps are isomorphisms. Hence (3.3) is surjective for $\mathcal{F}_2 = \underline{G}_2$ and any \mathcal{F}_1 .

To conclude, let \mathcal{F} be an extension of \mathcal{F}_2 by \mathcal{F}_1 in \mathcal{A} . By the above, $\mathcal{F}' := b_2^* \mathcal{F}$ is 1-motivic as an extension of \underline{G}_2 by \mathcal{F}_1 , and we have an exact sequence

$$0 \rightarrow L_2 \rightarrow \mathcal{F}' \rightarrow \mathcal{F} \rightarrow E_2 \rightarrow 0.$$

Let $b' : \underline{G} \rightarrow \mathcal{F}'$ be a normalised map (in the sense of Proposition 3.2.3) from a semi-abelian variety to \mathcal{F}' and let $b : \underline{G} \rightarrow \mathcal{F}$ be its composite with the above map. It is then an easy exercise to check that $\text{Ker } b$ and $\text{Coker } b$ are both discrete. Hence \mathcal{F} is 1-motivic. \square

3.3.2. Remark. We may similarly define 1-motivic sheaves for the fppf topology over $\text{Spec } k$; as one easily checks, all the above results hold equally well in this context. This is also the case for §3.7 below.

In fact, let $\text{Shv}_1^{\text{fppf}}$ be the $\mathbb{Z}[1/p]$ -linear category of fppf 1-motivic sheaves and $\pi : (\text{Spec } k)_{\text{fppf}} \rightarrow \text{Sm}(k)_{\text{ét}}$ be the projection functor. Then the functors π^* and π_* induce *quasi-inverse equivalences of categories* between Shv_1 and $\text{Shv}_1^{\text{fppf}}$. Indeed it suffices to check that $\pi_*\pi^*$ is naturally isomorphic to the identity on Shv_1 : if $\mathcal{F} \in \text{Shv}_1$ and we consider its normalised representation, then in the commutative diagram

$$\begin{array}{ccccccccc} 0 & \longrightarrow & L & \longrightarrow & \underline{G} & \longrightarrow & \mathcal{F} & \longrightarrow & E & \longrightarrow & 0 \\ & & \downarrow & & \downarrow & & \downarrow & & \downarrow & & \\ 0 & \longrightarrow & \pi_*\pi^*L & \longrightarrow & \pi_*\pi^*\underline{G} & \longrightarrow & \pi_*\pi^*\mathcal{F} & \longrightarrow & \pi_*\pi^*E & \longrightarrow & 0 \end{array}$$

the first, second and fourth vertical maps are isomorphisms and the lower sequence is still exact: both facts follow from [62, p. 14, Th. III.3.9].

In particular the restriction of π_* to $\text{Shv}_1^{\text{fppf}}$ is exact. Actually, $(R^q\pi_*)|_{\text{Shv}_1^{\text{fppf}}} = 0$ for $q > 0$ (use same reference).

3.4. A basic example.

3.4.1. Proposition. *Let $X \in \text{Sm}(k)$. Then the sheaf $\text{Pic}_{X/k}$ is 1-motivic.*

Proof. Suppose first that X is smooth projective. Then $\text{Pic}_{X/k}$ is an extension of the discrete sheaf $\text{NS}_{X/k}$ (Néron-Severi) by the abelian variety $\text{Pic}_{X/k}^0$ (Picard variety).

In general, we apply de Jong's theorem [47, Th. 4.1]: there exists a diagram

$$\begin{array}{ccc} \tilde{U} & \longrightarrow & \overline{X} \\ p \downarrow & & \\ U & \longrightarrow & X \end{array}$$

where the horizontal maps are open immersions, \overline{X} is smooth projective and the vertical map is finite étale. Then we get a corresponding

diagram of Pics

$$\begin{array}{ccc} \text{Pic}_{\tilde{U}/k} & \longleftarrow & \text{Pic}_{\overline{X}/k} \\ p^* \uparrow & & \\ \text{Pic}_{U/k} & \longleftarrow & \text{Pic}_{X/k} . \end{array}$$

The horizontal morphisms are epimorphisms and their kernels are lattices. This already shows by Proposition 3.2.3 d) that $\text{Pic}_{\tilde{U}/k} \in \text{Shv}_1$.

Consider the Čech spectral sequence associated to the étale cover p . It yields an exact sequence

$$\begin{aligned} 0 \rightarrow \check{H}^1(p, \underline{H}_{\text{ét}}^0(\tilde{U}, \mathbb{G}_m)) &\rightarrow \text{Pic}_{U/k} \rightarrow \check{H}^0(p, \text{Pic}_{\tilde{U}/k}) \\ &\rightarrow \check{H}^2(p, \underline{H}_{\text{ét}}^0(\tilde{U}, \mathbb{G}_m)). \end{aligned}$$

All the \check{H}^i are cohomology sheaves of complexes of objects of the abelian category Shv_1 , hence belong to Shv_1 ; it then follows from Theorem 3.3.1 that $\text{Pic}_{U/k} \in \text{Shv}_1$, as well as $\text{Pic}_{X/k}$. \square

3.5. Application: the Néron-Severi group of a smooth scheme.

3.5.1. Definition. Let $X \in \text{Sm}(k)$.

- a) Suppose that k is algebraically closed. Then we write $\text{NS}(X)$ for the group of cycles of codimension 1 on X modulo algebraic equivalence.
- b) In general, we define $\text{NS}_{X/k}$ as the étale sheaf on $\text{Sm}(k)$ given by

$$\text{NS}_{X/k}(U) = \text{NS}(X \times_k \overline{k(U)})^G$$

where $U \in \text{Sm}(k)$ is irreducible, $\overline{k(U)}$ is a separable closure of $k(U)$ and $G = \text{Gal}(\overline{k(U)}/k(U))$.

3.5.2. Proposition. *The natural map $e : \underline{\text{Pic}}_{X/k} \rightarrow \text{NS}_{X/k}$ identifies $\text{NS}_{X/k}$ with $\pi_0(\underline{\text{Pic}}_{X/k})$ (cf. Theorem 3.2.4 b)). In particular, $\text{NS}_{X/k} \in \text{Shv}_0$.*

Proof. It is well-known that cycles modulo algebraic equivalence are invariant by extension of algebraically closed base field. By Proposition 3.6.2 b), this implies that e induces a map $\bar{e} : \pi_0(\underline{\text{Pic}}_{X/k}) \rightarrow \text{NS}_{X/k}$, which is evidently epi. But let $\underline{\text{Pic}}_{X/k}^0 = \text{Ker } e$: by [17, Lemma 7.10], $\text{Pic}^0(X_{\bar{k}}) = \underline{\text{Pic}}_{X/k}^0(\bar{k})$ is divisible, which forces \bar{e} to be an isomorphism. \square

3.5.3. Remark. In particular, $\text{NS}(X)$ is finitely generated if k is algebraically closed: this was proven in [49, Th. 3] in a quite different way.

3.6. Technical results on 1-motivic sheaves.

3.6.1. **Proposition.** *The functor*

$$\begin{aligned} ev : \mathrm{Shv}_1 &\rightarrow \mathrm{Ab} \\ \mathcal{F} &\mapsto \mathcal{F}(\bar{k}) \end{aligned}$$

to the category Ab of abelian groups is exact and faithful, hence (cf. [20, Ch. 1, p. 44, prop. 1]) “faithfully exact”: a sequence \mathcal{E} is exact if and only if $ev(\mathcal{E})$ is exact.

Proof. The exactness of ev is clear. For faithfulness, let $\varphi : \mathcal{F}_1 \rightarrow \mathcal{F}_2$ be such that $ev(\varphi) = 0$. In $ev(3.2)$, we have $\varphi_G(\underline{G}_1(\bar{k})) \subseteq L_2(\bar{k})$; since the former group is divisible and the latter is finitely generated, $ev(\varphi_G) = 0$. Hence $\varphi_G = 0$. On the other hand, $ev(\varphi_E) = 0$, hence $\varphi_E = 0$. This implies that φ is of the form ψc_1 for $\psi : E_1 \rightarrow \mathcal{F}_2$. But $ev(\psi) = 0$, which implies that $\psi = 0$. \square

The following strengthens Theorem 3.2.4 b):

3.6.2. **Proposition.** a) *Let G be a commutative algebraic k -group and let E be a $\mathrm{Gal}(\bar{k}/k)$ -module, viewed as an étale sheaf over $\mathrm{Sm}(k)$ (E is not supposed to be constructible). Then $\mathrm{Hom}(\underline{G}, E) = 0$.*
b) *Let $\mathcal{F} \in \mathrm{Shv}_1$ and E as in a). Then any morphism $\mathcal{F} \rightarrow E$ factors canonically through $\pi_0(\mathcal{F})$.*

Proof. a) Thanks to Proposition 3.6.1 we may assume k algebraically closed. By Yoneda, $\mathrm{Hom}(\underline{G}, E)$ is a subgroup of $E(G)$ (it turns out to be the subgroup of multiplicative sections but we don’t need this). Since $E(k) \xrightarrow{\sim} E(G)$, any homomorphism from \underline{G} to E is constant, hence 0.

b) follows immediately from a) and Proposition 3.2.3. \square

3.6.3. **Lemma.** *Let $\mathcal{F} \in \mathrm{Shv}_1$, K a separably closed extension of k and M/K an algebraic extension. Then the map $\mathcal{F}(K) \rightarrow \mathcal{F}(M)$ is injective.*

Proof. Consider a normalised representation of \mathcal{F} :

$$(3.5) \quad 0 \rightarrow L \rightarrow \underline{G} \xrightarrow{b} \mathcal{F} \rightarrow E \rightarrow 0.$$

The lemma then follows from an elementary chase in the diagram:

$$\begin{array}{ccccccccc} 0 & \longrightarrow & L(K) & \longrightarrow & G(K) & \longrightarrow & \mathcal{F}(K) & \longrightarrow & E(K) & \longrightarrow & 0 \\ & & \wr \downarrow & & \text{mono} \downarrow & & \downarrow & & \wr \downarrow & & \\ 0 & \longrightarrow & L(M) & \longrightarrow & G(M) & \longrightarrow & \mathcal{F}(M) & \longrightarrow & E(M) & \longrightarrow & 0. \end{array}$$

\square

3.6.4. Definition. We denote by ${}^t\text{AbS}(k) = {}^t\text{AbS}$ the $\mathbb{Z}[1/p]$ -linear category of commutative k -group schemes G such that G^0 is semi-abelian and $\pi_0(G)$ is discrete. An object of ${}^t\text{AbS}$ is called a *semi-abelian scheme with torsion*.

3.6.5. Proposition. *The functor*

$$\begin{aligned} {}^t\text{AbS} &\rightarrow \text{Shv}_1 \\ G &\mapsto \underline{G} \end{aligned}$$

has a left adjoint/left inverse Ω .

Proof. Let $\mathcal{F} \in \text{Shv}_1$ with normalised representation (3.5). As the set of closed subgroups of $H \subseteq G$ is Artinian, there is a minimal H such that the composition

$$L \rightarrow \underline{G} \rightarrow \underline{G}/\underline{H}$$

is trivial. Then $\mathcal{F}/b(\underline{H})$ represents an object $\Omega(\mathcal{F})$ of ${}^t\text{AbS}$ and it follows from Proposition 3.2.3 b) that the universal property is satisfied. (In other words, $\Omega(\mathcal{F})$ is the quotient of \mathcal{F} by the Zariski closure of L in G .) \square

3.6.6. Proposition. *Let $f : \mathcal{F}_1 \rightarrow \mathcal{F}_2$ be a morphism in Shv_1 . Assume that for any $n > 1$ prime to p , f is an isomorphism on n -torsion and injective on n -cotorsion. Then f is injective with lattice cokernel. If f is even bijective on n -cotorsion, it is an isomorphism.*

Proof. a) We first treat the special case where $\mathcal{F}_1 = 0$. Consider multiplication by n on the normalised presentation of \mathcal{F}_2 :

$$\begin{array}{ccccccccc} 0 & \longrightarrow & L & \longrightarrow & G & \longrightarrow & \mathcal{F}_2 & \longrightarrow & E & \longrightarrow & 0 \\ & & n_L \downarrow & & n_G \downarrow & & n \downarrow & & n_E \downarrow & & \\ 0 & \longrightarrow & L & \longrightarrow & G & \longrightarrow & \mathcal{F}_2 & \longrightarrow & E & \longrightarrow & 0. \end{array}$$

Since L is torsion-free, n_G is injective for all n , hence $G = 0$ and $\mathcal{F}_2 = E$. If moreover multiplication by n is surjective for any n , we have $\mathcal{F}_2 = 0$ since E is finitely generated.

b) The general case. Split f into two short exact sequences:

$$\begin{aligned} 0 &\rightarrow K \rightarrow \mathcal{F}_1 \rightarrow I \rightarrow 0 \\ 0 &\rightarrow I \rightarrow \mathcal{F}_2 \rightarrow C \rightarrow 0. \end{aligned}$$

We get torsion/cotorsion exact sequences

$$\begin{aligned} 0 &\rightarrow {}_nK \rightarrow {}_n\mathcal{F}_1 \rightarrow {}_nI \rightarrow K/n \rightarrow \mathcal{F}_1/n \rightarrow I/n \rightarrow 0 \\ 0 &\rightarrow {}_nI \rightarrow {}_n\mathcal{F}_2 \rightarrow {}_nC \rightarrow I/n \rightarrow \mathcal{F}_2/n \rightarrow C/n \rightarrow 0. \end{aligned}$$

A standard diagram chase successively yields ${}_nK = 0$, ${}_n\mathcal{F}_1 \xrightarrow{\sim} {}_nI \xrightarrow{\sim} {}_n\mathcal{F}_2$, $\mathcal{F}_1/n \xrightarrow{\sim} I/n$, $K/n = 0$ and ${}_nC = 0$. By a), we find $K = 0$ and C a lattice, which is what we wanted. \square

3.7. Presenting 1-motivic sheaves by group schemes. In this subsection, we give another description of the category Shv_1 ; it will be used in the next subsection.

3.7.1. Definition. Let AbS be the $\mathbb{Z}[1/p]$ -linear category of commutative k -group schemes G such that $\pi_0(G)$ is a lattice and G^0 is a semi-abelian variety (it is a full subcategory of ${}^t\mathrm{AbS}$). We denote by S_1^{eff} the full subcategory of ${}^t\mathrm{AbS}^{[-1,0]}$ consisting of those complexes $F. = [F_1 \rightarrow F_0]$ such that

- (i) F_1 is discrete (*i.e.* in ${}^t\mathcal{M}_0$);
- (ii) F_0 is of the form $L_0 \oplus G$, with $L_0 \in {}^t\mathcal{M}_0$ and $G \in \mathrm{SAb}$;
- (iii) $F_1 \rightarrow F_0$ is a monomorphism;
- (iv) $\mathrm{Ker}(F_1 \rightarrow L_0)$ is free.

We call S_1^{eff} the *category of presentations*.

We shall view S_1^{eff} as a full subcategory of $\mathrm{Shv}_1^{[-1,0]}$ via the functor $G \mapsto \underline{G}$ which sends a group scheme to the associated representable sheaf. In this light, $F.$ may be viewed as a *presentation* of $\mathcal{F} := H_0(\underline{F.})$. In the next definition, quasi-isomorphisms are also understood from this viewpoint.

3.7.2. Definition. We denote by Σ the collection of quasi-isomorphisms of S_1^{eff} , by $\overline{S}_1^{\mathrm{eff}}$ the homotopy category of S_1^{eff} (Hom groups quotiented by homotopies) and by $S_1 = \Sigma^{-1}\overline{S}_1^{\mathrm{eff}}$ the localisation of $\overline{S}_1^{\mathrm{eff}}$ with respect to (the image of) Σ .

The functor $F. \mapsto H_0(F.)$ induces a functor

$$(3.6) \quad h_0 : S_1 \rightarrow \mathrm{Shv}_1.$$

Let $F. = (F_1, L_0, G)$ be a presentation of $\mathcal{F} \in \mathrm{Shv}_1$. Let $L = \mathrm{Ker}(F_1 \rightarrow L_0)$ and $E = \mathrm{Coker}(F_1 \rightarrow L_0)$. Then we clearly have an exact sequence

$$(3.7) \quad 0 \rightarrow L \rightarrow \underline{G} \rightarrow \mathcal{F} \rightarrow E \rightarrow 0.$$

3.7.3. Lemma. *Let $F. = (F_1, L_0, G) \in S_1^{\mathrm{eff}}$. Then, for any finite Galois extension ℓ/k such that L_0 is constant over ℓ , there exists a q.i. $\tilde{F}. \rightarrow F.$, with $\tilde{F}. = [\tilde{F}_1 \xrightarrow{u_0} \tilde{L}_0 \oplus G]$ such that u_0 is diagonal and \tilde{L}_0 is a free $\mathrm{Gal}(\ell/k)$ -module.*

Proof. Just take for \tilde{L}_0 a free module projecting onto L_0 and for $\tilde{F}_1 \rightarrow \tilde{L}_0$ the pull-back of $F_1 \rightarrow L_0$. \square

3.7.4. Lemma. *The set Σ admits a calculus of right fractions within $\overline{S}_1^{\text{eff}}$ in the sense of (the dual of) [35, Ch. I, §2.3].*

Proof. The statement is true by Lemma A.2.2 if we replace S_1^{eff} by $\text{Shv}_1^{[-1,0]}$; but one easily checks that the constructions in the proof of Lemma A.2.2 preserve S_1^{eff} . \square

3.7.5. Proposition. *The functor h_0 of (3.6) is an equivalence of categories. In particular, S_1 is abelian.*

Proof. Step 1. h_0 is essentially surjective. Let $\mathcal{F} \in \text{Shv}_1$ and let (3.7) be the exact sequence attached to it by Proposition 3.2.3 b). We shall construct a presentation of \mathcal{F} from (3.7). Choose elements $f_1, \dots, f_r \in \mathcal{F}(\bar{k})$ whose images generate $E(\bar{k})$. Let ℓ/k be a finite Galois extension such that all f_i belong to $\mathcal{F}(\ell)$, and let $\Gamma = \text{Gal}(\ell/k)$. Let $\tilde{L}_0 = \mathbb{Z}[\Gamma]^r$ and define a morphism of sheaves $\tilde{L}_0 \rightarrow \mathcal{F}$ by mapping the i -th basis element to f_i . Then $\text{Ker}(\tilde{L}_0 \rightarrow E)$ maps to \underline{G}/L . Let M_0 be the kernel of this morphism, and let $L_0 = \tilde{L}_0/M_0$. Then $\tilde{L}_0 \twoheadrightarrow E$ factors into a morphism $L_0 \twoheadrightarrow E$, whose kernel K injects into \underline{G}/L .

Pick now elements $g_1, \dots, g_s \in G(\bar{k})$ whose image in $G(\bar{k})/L(\bar{k})$ generate the image of $K(\bar{k})$, and $g_{s+1}, \dots, g_t \in G(\bar{k})$ be generators of the image of $L(\bar{k})$. Let ℓ'/k be a finite Galois extension such that all the g_i belong to $G(\ell')$, and let $\Gamma' = \text{Gal}(\ell'/k)$. Let $\tilde{F}_1 = \mathbb{Z}[\Gamma']^t$, and define a map $f : \tilde{F}_1 \rightarrow G$ by mapping the i -th basis element to g_i . By construction, $f^{-1}(L) = \text{Ker}(\tilde{F}_1 \twoheadrightarrow K)$ and $f' : f^{-1}(L) \rightarrow L$ is onto. Let M_1 be the kernel of f' and $F_1 = \tilde{F}_1/M_1$; then $\tilde{F}_1 \rightarrow K$ factors through F_1 and $\text{Ker}(F_1 \twoheadrightarrow K) = \text{Ker}(F_1 \rightarrow L_0) \xrightarrow{\sim} L$. In particular, condition (iii) of Definition 3.7.1 is verified.

Step 2. h_0 is faithful. Let $f : F \rightarrow F'$ be a map in S_1 such that $h_0(f) = 0$. By Lemma 3.7.4, we may assume that f is an effective map (i.e. comes from S_1^{eff}). We have $f(L_0 \oplus G) \subseteq \text{Im}(L'_1 \rightarrow L'_0 \oplus G')$, hence $f|_G = 0$ and $f(L_0)$ is contained in $\text{Im}(L'_1 \rightarrow L'_0 \oplus G')$. Pick a finite Galois extension ℓ/k such that L_0 and L'_1 are constant over ℓ . By Lemma 3.7.3, take a q.i. $u : [\tilde{F}_1 \rightarrow \tilde{L}_0] \rightarrow [F_1 \rightarrow L_0]$ such that \tilde{L}_0 is $\text{Gal}(\ell/k)$ -free. Then the composition $\tilde{L}_0 \rightarrow L_0 \rightarrow \text{Im}(L'_1 \rightarrow L'_0 \oplus G')$ lifts to a map $s : \tilde{L}_0 \rightarrow L'_1$, which defines a homotopy between 0 and fu .

Step 3. h_0 is full. Let $F, F' \in S_1$ and let $\varphi : \mathcal{F} \rightarrow \mathcal{F}'$, where $\mathcal{F} = h_0(F)$ and $\mathcal{F}' = h_0(F')$. In particular, we get a map $\varphi_G : \underline{G} \rightarrow \underline{G}'$

and a map $\psi : L_0 \rightarrow L'_0 \oplus \underline{G}'/F'_1$. Let ℓ/k be a finite Galois extension such that F'_1 is constant over ℓ . Pick a q.i. $u : \tilde{F} \rightarrow F$ as in Lemma 3.7.3 such that \tilde{L}_0 is $\text{Gal}(\ell/k)$ -free. Then $\psi \circ u$ lifts to a map $\tilde{\psi} : \tilde{L}_0 \rightarrow L'_0 \oplus \underline{G}'$. The map

$$f = (\tilde{\psi}, \varphi_G) : \tilde{L}_0 \oplus G \rightarrow L'_0 \oplus G'$$

sends \tilde{F}_1 into F'_1 by construction, hence yields a map $f : \tilde{F} \rightarrow F'$ such that $h_0(fu^{-1}) = \varphi$. \square

3.7.6. Corollary. *The obvious functor*

$$S_1 \rightarrow D^b(\text{Shv}_1)$$

is fully faithful.

Proof. The composition of this functor with H_0 is the equivalence h_0 of Proposition 3.7.5. Therefore it suffices to show that the restriction of H_0 to the image of S_1 is faithful. This is obvious, since the objects of this image are homologically concentrated in degree 0. \square

3.8. The transfer structure on 1-motivic sheaves. Recall the category AbS from Definition 3.7.1. Lemma 1.3.2 provides a functor

$$\rho : \text{AbS} \rightarrow \text{HI}_{\text{ét}}[1/p].$$

The aim of this subsection is to prove:

3.8.1. Proposition. *This functor extends to a full embedding*

$$\rho : \text{Shv}_1 \hookrightarrow \text{HI}_{\text{ét}}[1/p]$$

where $\text{HI}_{\text{ét}}$ is the category of Definitions 1.3.1 and D.1.2. This functor is exact with thick image (i.e. stable under extensions).

Proof. By Proposition 3.7.5, it suffices to construct a functor $\rho : S_1 \rightarrow \text{HI}_{\text{ét}}[1/p]$. First define a functor $\tilde{\rho} : S_1^{\text{eff}} \rightarrow \text{HI}_{\text{ét}}[1/p]$ by

$$\tilde{\rho}([F_1 \rightarrow F_0]) = \text{Coker}(\rho(F_1) \rightarrow \rho(F_0)).$$

Note that the forgetful functor $f : \text{HI}_{\text{ét}} \rightarrow \text{Shv}_{\text{ét}}(\text{Sm}(k))$ is faithful and exact, hence conservative. This first gives that $\tilde{\rho}$ factors into the desired ρ .

Proposition 3.2.3 d) says that $f\rho$ is (fully faithful and) exact. Since f is faithful, ρ is fully faithful and exact.

It remains to show that ρ is thick. Recall that Shv_1 is thick in $\text{Shv}_{\text{ét}}(\text{Sm}(k))[1/p]$ by Theorem 3.3.1. Since f is exact, we are then left to show:

3.8.2. Lemma. *The transfer structure on a sheaf $\mathcal{F} \in \text{Shv}_1$ is unique.*

Proof. Let μ be the transfer structure on \mathcal{F} given by the beginning of the proof of Proposition 3.8.1, and let μ' be another transfer structure. Thus, for $X, Y \in Sm(k)$, we have two homomorphisms

$$\mu, \mu' : \mathcal{F}(X) \otimes c(Y, X) \rightarrow \mathcal{F}(Y)$$

and we want to show that they are equal. We may clearly assume that Y is irreducible.

Let $F = k(Y)$ be the function field of Y . Since \mathcal{F} is a homotopy invariant Zariski sheaf with transfers, the map $\mathcal{F}(Y) \rightarrow \mathcal{F}(F)$ is injective by [88, Cor. 4.19]. Thus we may replace Y by F .

Moreover, it follows from the fact that \mathcal{F} is an étale sheaf and from Lemma 3.6.3 that $\mathcal{F}(F) \hookrightarrow \mathcal{F}(\overline{F})$, where \overline{F} is an algebraic closure of F . Thus, we may even replace Y by \overline{F} .

Then the group $c(Y, X)$ is replaced by $c(\overline{F}, X) = Z_0(X_{\overline{F}})$. Since \overline{F} is algebraically closed, all closed points of $X_{\overline{F}}$ are rational, hence all finite correspondences from $\text{Spec } \overline{F}$ to X are linear combinations of morphisms. Therefore μ and μ' coincide on them. \square

This concludes the proof of Proposition 3.8.1. \square

3.9. 1-motivic sheaves and DM. Recall from Definition D.1.2 the subcategory $\text{HI}_{\text{ét}}^s \subset \text{HI}_{\text{ét}}$ of strictly homotopy invariant étale sheaves with transfers: this is a full subcategory of $\text{DM}_{-, \text{ét}}^{\text{eff}}$. By Proposition D.1.4, we have

$$\text{HI}_{\text{ét}}^s = \{\mathcal{F} \otimes_{\mathbb{Z}} \mathbb{Z}[1/p] \mid \mathcal{F} \in \text{HI}_{\text{ét}}\}.$$

The introduction of Shv_1 is now made clear by the following

3.9.1. Theorem. *Let $\text{Shv}_1^s \subseteq \text{HI}_{\text{ét}}^s$ be the full subcategory image of Shv_1 by the functor $\mathcal{F} \mapsto \mathcal{F}[1/p]$ of Lemma D.1.3. Then $\text{Shv}_1 \rightarrow \text{Shv}_1^s$ is an equivalence of categories. Moreover, let $M \in d_{\leq 1} \text{DM}_{\text{gm}, \text{ét}}^{\text{eff}}$. Then for all $i \in \mathbb{Z}$, $\mathcal{H}_i(M) \in \text{Shv}_1^s$. In particular, there is a t -structure on $d_{\leq 1} \text{DM}_{\text{gm}, \text{ét}}^{\text{eff}}$, with heart Shv_1^s ; it is induced by the homotopy t -structure of Corollary D.3.3 on $\text{DM}_{\text{fr}, \text{ét}}^{\text{eff}}$ (see Definition D.2.1 and Theorem D.2.2).*

Proof. The first assertion is clear since $\text{HI}_{\text{ét}}^s \rightarrow \text{HI}_{\text{ét}} \rightarrow \text{HI}_{\text{ét}}[1/p]$ is an equivalence of categories by Proposition D.1.4. By Proposition 3.2.3 d), we reduce to the case $M = M(C)$, $C \xrightarrow{p} \text{Spec } k$ a smooth projective curve. By Proposition 2.5.2, the cohomology sheaves of $M(C)$ belong to Shv_1^s : for \mathcal{H}_1 this is clear and for \mathcal{H}_2 it is a (trivial) special case of Proposition 3.4.1. \square

Note that the functor $\mathcal{M}_1 \rightarrow \mathrm{HI}_{\mathrm{\acute{e}t}}^{[0,1]}$ refines to an functor $\mathcal{M}_1 \rightarrow \mathrm{Shv}_1^{[0,1]}$, hence, using Lemma A.2.1 again, we get a composed triangulated functor

$$(3.8) \quad \mathrm{tot} : D^b(\mathcal{M}_1[1/p]) \rightarrow D^b(\mathrm{Shv}_1^{[0,1]}) \rightarrow D^b(\mathrm{Shv}_1)$$

refining the one from Lemma 2.2.1 (same proof). We then get:

3.9.2. Corollary. *The two functors*

$$D^b(\mathcal{M}_1[1/p]) \xrightarrow{\mathrm{tot}} D^b(\mathrm{Shv}_1) \rightarrow d_{\leq 1} \mathrm{DM}_{\mathrm{gm}, \mathrm{\acute{e}t}}^{\mathrm{eff}}$$

are equivalences of categories.

Proof. For the composition, this is Theorem 2.1.2. This implies that the second functor is full and essentially surjective, and to conclude, it suffices by Lemma A.1.1 to see that it is conservative. But this follows immediately from Theorem 3.9.1. Note that by Proposition D.1.4 we have that $\mathrm{Shv}_1^s \simeq \mathrm{Shv}_1$ (see Theorem 3.9.1 for the definition of Shv_1^s). \square

3.9.3. Definition. We call the t -structure defined on $D^b(\mathcal{M}_1[1/p])$ or on $d_{\leq 1} \mathrm{DM}_{\mathrm{gm}, \mathrm{\acute{e}t}}^{\mathrm{eff}}$ by Corollary 3.9.2 the *homotopy t -structure*.

3.9.4. Remark (*cf.* §1.9). In [11], A. Bertapelle defines an *integral* version of the category $\mathrm{Shv}_1^{\mathrm{fppf}}$ of Remark 3.3.2 and constructs an equivalence of categories

$$D^b(\mathcal{M}_1) \simeq D^b(\mathrm{Shv}_1^{\mathrm{fppf}})$$

without inverting p (not going via DM). Hence the homotopy t -structure of Definition 3.9.3 exists integrally.

3.10. Comparing t -structures. In this subsection, we want to compare the homotopy t -structure of Definition 3.9.3 with the motivic t -structure of Theorem 1.6.1 a).

Let $C \in D^b(\mathcal{M}_1[1/p])$. Recall from 1.8.7 the notation ${}^t H_n(C) \in {}^t \mathcal{M}_1[1/p]$ for its homology relative to the torsion 1-motivic t -structure from Theorem 1.6.1. We also write $\mathcal{H}^n(C) \in \mathrm{Shv}_1$ for its cohomology objects of relative to the homotopy t -structure.

Consider the functor tot of (3.8). Let \mathcal{F} be a 1-motivic sheaf and (G, b) its associated normalised pair (see Proposition 3.2.3 a)). Let $L = \mathrm{Ker} b$ and $E = \mathrm{Coker} b$. In $D^b(\mathcal{M}_1[1/p])$, we have an exact triangle

$$[L \rightarrow G][1] \rightarrow \mathrm{tot}^{-1}(\mathcal{F}) \rightarrow [E \rightarrow 0] \xrightarrow{+1}$$

(see Corollary 3.9.2). This shows:

3.10.1. **Lemma.** *We have*

$$\begin{aligned} {}^tH_0(\mathrm{tot}^{-1}(\mathcal{F})) &= [E \rightarrow 0] \\ {}^tH_1(\mathrm{tot}^{-1}(\mathcal{F})) &= [L \rightarrow G] \\ {}^tH_q(\mathrm{tot}^{-1}(\mathcal{F})) &= 0 \text{ for } q \neq 0, 1. \quad \square \end{aligned}$$

On the other hand, given a 1-motive (with torsion or cotorsion) $M = [L \xrightarrow{f} G]$, we clearly have

$$\begin{aligned} \mathcal{H}^0(M) &= \mathrm{Ker} f \\ \mathcal{H}^1(M) &= \mathrm{Coker} f \\ \mathcal{H}^q(M) &= 0 \text{ for } q \neq 0, 1. \end{aligned} \tag{3.9}$$

by considering it as a complex of length 1 of 1-motivic sheaves.

In particular, ${}^t\mathcal{M}_1[1/p] \cap \mathrm{Shv}_1 = \mathrm{Shv}_0$, ${}^t\mathcal{M}_1[1/p] \cap \mathrm{Shv}_1[-1]$ consists of quotients of semi-abelian varieties by discrete subsheaves and ${}^t\mathcal{M}_1[1/p] \cap \mathrm{Shv}_1[q] = 0$ for $q \neq 0, -1$.

Here is a more useful result relating \mathcal{H}^i with the two motivic t -structures:

3.10.2. **Proposition.** *Let $C \in D^b(\mathcal{M}_1[1/p])$; write $[L_i \xrightarrow{u_i} G_i]$ for ${}^tH_i(C)$ and $[L^i \xrightarrow{u^i} G^i]$ for ${}^tH^i(C)$ ⁷. Then we have exact sequences in Shv_1 :*

$$\begin{aligned} \cdots \rightarrow L_{i+1} \xrightarrow{u_{i+1}} G_{i+1} \rightarrow \mathcal{H}_i(C) \rightarrow L_i \xrightarrow{u_i} G_i \rightarrow \cdots \\ \cdots \rightarrow L^{i-1} \xrightarrow{u^{i-1}} G^{i-1} \rightarrow \mathcal{H}^i(C) \rightarrow L^i \xrightarrow{u^i} G^i \rightarrow \cdots \end{aligned}$$

Proof. For the first one, argue by induction on the length of C with respect to the motivic t -structure with heart ${}^t\mathcal{M}_1[1/p]$ (the case of length 0 is (3.9)). For the second one, same argument with the other motivic t -structure. \square

Note finally that the homotopy t -structure is far from being invariant under Cartier duality: this can easily be seen by using Proposition 3.7.5.

4. COMPARING TWO DUALITIES

In this section, we show that the classical Cartier duality for 1-motives is compatible with a “motivic Cartier duality” on triangulated motives, described in Definition 4.5.2 below.

⁷Note that (L_i, G_i) and (L^i, G^i) are determined only up to the relevant q.i.’s.

4.1. Biextensions of 1-motives. This material is presumably well-known to experts, and the only reason why we write it up is that we could not find it in the literature. Exceptionally, we put 1-motives in degrees -1 and 0 in this subsection and in the next one, for compatibility with Deligne's conventions in [25].

Recall (see [25, §10.2]) that for $M_1 = [L_1 \xrightarrow{u_1} G_1]$ and $M_2 = [L_2 \xrightarrow{u_2} G_2]$ two complexes of abelian sheaves over some site \mathcal{S} , concentrated in degrees -1 and 0 , a *biextension* of M_1 and M_2 by an abelian sheaf H is given by a (Grothendieck) biextension P of G_1 and G_2 by H and a pair of compatible trivializations of the biextensions of $L_1 \times G_2$ and $G_1 \times L_2$ obtained by pullbacks. Let $\text{Biext}(M_1, M_2; H)$ denote the group of isomorphism classes of biextensions. We have the following fundamental formula (see [25, §10.2.1]):

$$(4.1) \quad \text{Biext}(M_1, M_2; H) = \text{Ext}_{\mathcal{S}}^1(M_1 \overset{L}{\otimes} M_2, H).$$

Suppose now that M_1 and M_2 are two Deligne 1-motives. Since G_1 and G_2 are smooth, we may compute biextensions by using the étale topology. Hence, we shall take here

$$\mathcal{S} = \text{Sm}(k)_{\text{ét}}.$$

Let M_2^* denote the Cartier dual of M_2 as constructed by Deligne (see [25, §10.2.11] and [8, §0]) along with the Poincaré biextension $P_{M_2} \in \text{Biext}(M_2, M_2^*; \mathbb{G}_m)$. We also have the transpose ${}^t P_{M_2} = P_{M_2^*} \in \text{Biext}(M_2^*, M_2; \mathbb{G}_m)$. Pulling back ${}^t P_{M_2}$ yields a map

$$(4.2) \quad \gamma_{M_1, M_2} : \text{Hom}(M_1, M_2^*) \rightarrow \text{Biext}(M_1, M_2; \mathbb{G}_m)$$

$$\varphi \mapsto (\varphi \times 1_{M_1})^*({}^t P_{M_2})$$

which is clearly additive and natural in M_1 .

4.1.1. Proposition. *The map γ_{M_1, M_2} yields a natural equivalence of functors from 1-motives to abelian groups, i.e. the functor*

$$M_1 \mapsto \text{Biext}(M_1, M_2; \mathbb{G}_m)$$

on 1-motives is representable by the Cartier dual M_2^ . Moreover, γ_{M_1, M_2} is also natural in M_2 .*

Proof. We start with a few lemmas:

4.1.2. Lemma. *For $q \leq 0$, we have*

$$\text{Hom}_{\overline{k}}(M_1 \overset{L}{\otimes} M_2, \mathbb{G}_m[q]) = 0.$$

Proof. For $q < 0$ this is trivial and for $q = 0$ this is [25, Lemma 10.2.2.1]. \square

4.1.3. Lemma. *Let \bar{k} be an algebraic closure of k and $G = \text{Gal}(\bar{k}/k)$. Then*

$$\begin{aligned} \text{Hom}_k(M_1, M_2^*) &\xrightarrow{\sim} \text{Hom}_{\bar{k}}(M_1, M_2^*)^G \\ \text{Biext}_k(M_1, M_2; \mathbb{G}_m) &\xrightarrow{\sim} \text{Biext}_{\bar{k}}(M_1, M_2; \mathbb{G}_m)^G. \end{aligned}$$

Proof. The first isomorphism is obvious. For the second, thanks to (4.1) we may use the spectral sequence

$$H^p(G, \text{Hom}_{\bar{k}}(M_1 \overset{L}{\otimes} M_2, \mathbb{G}_m[q])) \Rightarrow \text{Hom}_k(M_1 \overset{L}{\otimes} M_2, \mathbb{G}_m[p+q]).$$

(This is the only place in the proof of Proposition 4.1.1 where we shall use (4.1).) The assertion then follows from Lemma 4.1.2. \square

Lemma 4.1.3, reduces the proof of Proposition 4.1.1 to the case where k is algebraically closed, which we now assume. The following is a special case of this proposition:

4.1.4. Lemma. *The map γ_{M_1, M_2} is an isomorphism when M_1 and M_2 are abelian varieties A_1 and A_2 , and is natural in A_2 .*

Again this is certainly well-known and explicitly mentioned as such in [SGA7, VII, p. 176, (2.9.6.2)]. Unfortunately we have not been able to find a proof in the literature, so we provide one for the reader's convenience.

Proof. We shall use the universal property of the Poincaré bundle [65, Th. p. 125]. Let $P \in \text{Biext}(A_1, A_2)$. Then

- (1) $P|_{A_1 \times \{0\}}$ is trivial;
- (2) $P|_{\{a\} \times A_2} \in \text{Pic}^0(A_2)$ for all $a \in A_1(k)$.

Indeed, (1) follows from the multiplicativity of P on the A_2 -side. For (2) we offer two proofs (note that they use multiplicativity on different sides):

- By multiplicativity on the A_1 -side, $a \mapsto P|_{\{a\} \times A_2}$ gives a homomorphism $A_1(k) \rightarrow \text{Pic}(A_2)$. Composing with the projection to $\text{NS}(A_2)$ gives a homomorphism from a divisible group to a finitely generated group, which must be trivial.
- (More direct but more confusing): we have to prove that $T_b^* P|_{\{a\} \times A_2} = P|_{\{a\} \times A_2}$ for all $b \in A_2(k)$. Using simply a to

denote the section $\text{Spec } k \rightarrow A_1$ defined by a , we have a commutative diagram

$$\begin{array}{ccc} A_2 & \xrightarrow{a \times 1_{A_2}} & A_1 \times A_2 \\ T_b \downarrow & & \downarrow 1_{A_1} \times T_b \\ A_2 & \xrightarrow{a \times 1_{A_2}} & A_1 \times A_2. \end{array}$$

Let $\pi_1 : A_1 \rightarrow \text{Spec } k$ and $\pi_2 : A_2 \rightarrow \text{Spec } k$ be the two structural maps. Then by multiplicativity on the A_2 -side, an easy computation gives

$$(1_{A_1} \times T_b)^* P = P \otimes (1_{A_1} \times (\pi_2 \circ b))^* P.$$

Applying $(a \times 1_{A_2})^*$ to this gives the result since $(a \times 1_{A_2})^* \circ (1_{A_1} \times (\pi_2 \circ b))^* P = \pi_{A_2}^* P_{a,b}$ is trivial.

By the universal property of the Poincaré bundle, there exists a unique morphism⁸ $f : A_1 \rightarrow A'_2$ such that $P \simeq (f \times 1_{A_2})^*({}^t P_{A_2})$. It remains to see that f is a homomorphism: for this it suffices to show that $f(0) = 0$. But

$$\begin{aligned} \mathcal{O}_{A_2} &\simeq P_{\{0\} \times A_2} = (0 \times 1_{A_2})^* \circ (f \times 1_{A_2})^*({}^t P_{A_2}) \\ &= (f(0) \times 1_{A_2})^*({}^t P_{A_2}) = (P_{A_2})_{|A_2 \times \{f(0)\}} = f(0) \end{aligned}$$

where the first isomorphism holds by multiplicativity of P on the A_1 -side.

Finally, the naturality in A_2 reduces to the fact that, if $f : A_1 \rightarrow A'_2$, then $(f \times 1_{A_2})^*({}^t P_{A_2}) \simeq (1_{A_1} \times f')^*(P_{A_1})$. This follows from the description of f' on k -points as the pull-back by f of line bundles. \square

We also have the following easier

4.1.5. Lemma. *Let L be a lattice and A an abelian variety. Then the natural map*

$$\begin{aligned} \text{Hom}(L, A') &\rightarrow \text{Biext}(L[0], A[0]; \mathbb{G}_m) \\ f &\mapsto (1 \times f)^*({}^t P_A) \end{aligned}$$

is bijective.

Proof. Reduce to $L = \mathbb{Z}$; then the right hand side can be identified with $\text{Ext}(A, \mathbb{G}_m)$ and the claim comes from the Weil-Barsotti formula. \square

⁸For convenience we denote here by A' the dual of an abelian variety A and by f' the dual of a homomorphism f of abelian varieties.

Let us now come back to our two 1-motives M_1, M_2 . We denote by L_i, T_i and A_i the discrete, toric and abelian parts of M_i for $i = 1, 2$. Let us further denote by $u'_i : L'_i \rightarrow A'_i$ the map corresponding to G_i under the isomorphism $\text{Ext}(A_i, T_i) \simeq \text{Hom}(L'_i, A'_i)$ where $L'_i = \text{Hom}(T_i, \mathbb{G}_m)$ and $A'_i = \text{Pic}^0(A_i)$.

We shall use the symmetric avatar $(L_i, A_i, L'_i, A'_i, \psi_i)$ of M_i (see [25, 10.2.12] or [8, p. 17]): recall that ψ_i denotes a certain section of the Poincaré biextension $P_{A_i} \in \text{Biext}(A_i, A'_i; \mathbb{G}_m)$ over $L_i \times L'_i$. The symmetric avatar of the Cartier dual is $(L'_i, A'_i, L_i, A_i, \psi_i^t)$. By *loc. cit.* a map of 1-motives $\varphi : M_1 \rightarrow M_2^*$ is equivalent to a homomorphism $f : A_1 \rightarrow A'_2$ of abelian varieties and, if f' is the dual of f , liftings g and g' of fu_1 and $f'u_2$ respectively, *i.e.* to the following commutative squares

$$(4.3) \quad \begin{array}{ccc} L_1 & \xrightarrow{g} & L'_2 \\ u_1 \downarrow & & u'_2 \downarrow \\ A_1 & \xrightarrow{f} & A'_2 \end{array} \quad \text{and} \quad \begin{array}{ccc} L_2 & \xrightarrow{g'} & L'_1 \\ u_2 \downarrow & & u'_1 \downarrow \\ A_2 & \xrightarrow{f'} & A'_1 \end{array}$$

under the condition that

$$(4.4) \quad (1_{L_1} \times g')^* \psi_1 = (g \times 1_{L_2})^{*t} \psi_2 \text{ on } L_1 \times L_2.$$

Now let (P, τ, σ) be a biextension of M_1 and M_2 by \mathbb{G}_m , *i.e.* a biextension $P \in \text{Biext}(G_1, G_2; \mathbb{G}_m)$, a section τ on $L_1 \times G_2$ and a section σ on $G_1 \times L_2$ such that

$$(4.5) \quad \tau|_{L_1 \times L_2} = \sigma|_{L_1 \times L_2}.$$

We have to show that $(P, \tau, \sigma) = (\varphi \times 1)^*({}^t P_{A_2}, \tau_2, \sigma_2)$ for a unique $\varphi : M_1 \rightarrow M_2^*$, where τ_2 and σ_2 are the universal trivializations.

Recall that $\text{Biext}(G_1, G_2; \mathbb{G}_m) = \text{Biext}(A_1, A_2; \mathbb{G}_m)$ (*cf.* [25, 10.2.3.9]) so that, by Lemma 4.1.4, P is the pull-back to $G_1 \times G_2$ of $(f \times 1_{A_2})^*({}^t P_{A_2})$ for a unique homomorphism $f : A_1 \rightarrow A'_2$. We thus have obtained the map f and its dual f' in (4.3), and we now want to show that the extra data (τ, σ) come from a pair (g, g') in a unique way.

We may view $E = (fu_1 \times 1_{A_2})^*({}^tP_{A_2})$ as an extension of $L_1 \otimes A_2$ by \mathbb{G}_m . Consider the commutative diagram of exact sequences

$$\begin{array}{ccccccc}
 & 0 & & 0 & & 0 & \\
 & \downarrow & & \downarrow & & \downarrow & \\
 0 & \longrightarrow & 0 & \longrightarrow & L_1 \otimes T_2 & \xlongequal{\quad} & L_1 \otimes T_2 \longrightarrow 0 \\
 & & \downarrow & & \downarrow & & \downarrow 1_{L_1} \otimes i_2 \\
 (4.6) \quad 0 & \longrightarrow & \mathbb{G}_m & \xrightarrow{i} & Q & \xrightarrow{\pi'} & L_1 \otimes G_2 \longrightarrow 0 \\
 & & \parallel & & \downarrow \pi & & \downarrow 1_{L_1} \otimes p_2 \\
 0 & \longrightarrow & \mathbb{G}_m & \longrightarrow & E & \longrightarrow & L_1 \otimes A_2 \longrightarrow 0 \\
 & & \downarrow & & \downarrow & & \downarrow \\
 & & 0 & & 0 & & 0
 \end{array}$$

where i_2 (*resp.* p_2) is the inclusion $T_2 \hookrightarrow G_2$ (*resp.* the projection $G_2 \rightarrow A_2$). The section τ yields a retraction $\tilde{\tau} : Q \rightarrow \mathbb{G}_m$ whose restriction to $L_1 \otimes T_2$ yields a homomorphism

$$\tilde{g} : L_1 \otimes T_2 \rightarrow \mathbb{G}_m$$

which in turn defines a homomorphism as in (4.3). We denote the negative of this morphism by g .

4.1.6. Lemma. *With this choice of g , the left square of (4.3) commutes and $\tau = (g \times 1_{G_2})^*\tau_2$.*

Proof. To see the first assertion, we may apply $\text{Ext}^*(-, \mathbb{G}_m)$ to (4.6) and then apply [15, Lemma 2.8] to the corresponding diagram. Here is a concrete description of this argument: via the map of Lemma 4.1.5, u'_2g goes to the following pushout

$$\begin{array}{ccccccc}
 0 & \longrightarrow & L_1 \otimes T_2 & \xrightarrow{1 \otimes i_2} & L_1 \otimes G_2 & \xrightarrow{1 \otimes p_2} & L_1 \otimes A_2 \longrightarrow 0 \\
 & & \downarrow -\tilde{g} & & \downarrow \pi \circ \tau & & \downarrow \parallel \\
 0 & \longrightarrow & \mathbb{G}_m & \longrightarrow & E & \longrightarrow & L_1 \otimes A_2 \longrightarrow 0.
 \end{array}$$

because, due to the relation $i\tilde{\tau} + \tau\pi' = 1$, the left square in this diagram commutes.

In particular, we have

$$\begin{aligned}
 Q &= (1 \otimes p_2)^*(fu_1 \otimes 1)^{{}^t}P_{A_2} = (fu_1 \otimes p_2)^{{}^t}P_{A_2} \\
 &= (u'_2g \otimes p_2)^{{}^t}P_{A_2} = (g \otimes 1)^*(u'_2 \otimes p_2^*)^{{}^t}P_{A_2}.
 \end{aligned}$$

For the second assertion, since $\mathrm{Hom}(L_1 \otimes A_2, \mathbb{G}_m) = 0$ it suffices to check the equality after restricting to $L_1 \otimes T_2$. This is clear because under the isomorphism $\mathrm{Hom}(L'_2 \otimes T_2, \mathbb{G}_m) = \mathrm{Hom}(L'_2, L'_2)$, the canonical trivialization ${}^t\psi_2$ corresponds to the identity. \square

Note that if we further pullback we obtain that

$$(4.7) \quad \tau|_{L_1 \times L_2} = \psi_2^t|_{L_1 \times L_2}.$$

The same computation with σ yields a map

$$g' : L_2 \rightarrow L'_1$$

and the same argument as in Lemma 4.1.6 shows that with this choice of g' the right square of (4.3) commutes. We now use that $P = (1_{A_1} \times f')^*(P_{A_1})$, which follows from the naturality statement in Lemma 4.1.4. As in the proof of Lemma 4.1.6, this implies that its trivialization σ on $G_1 \times L_2$ is the pullback of the canonical trivialization ψ_1 on $G_1 \times L'_1$ along $1_{G_1} \times g' : G_1 \times L_2 \rightarrow G_1 \times L'_1$. In particular:

$$(4.8) \quad \sigma|_{L_1 \times L_2} = \psi_1|_{L_1 \times L_2}.$$

Put together, (4.5), (4.7) and (4.8) show that Condition (4.4) is verified: thus we get a morphism $\varphi : M_1 \rightarrow M_2^*$. Let $h : G_1 \rightarrow G'_2$ be its group component. It remains to check that $\sigma = (h \times 1_{L_2})^*\sigma_2$. As in the proof of Lemma 4.1.6 we only need to check this after restriction to $T_1 \otimes L_2$. But the restriction of h to the toric parts is the Cartier dual of g' , so we conclude by the same argument.

Finally, let us show that γ_{M_1, M_2} is natural in M_2 . This amounts to comparing two biextensions. For the bitorsors this follows from Lemma 4.1.4 and for the sections we may argue again as in the proof of Lemma 4.1.6. \square

4.2. Biextensions of complexes of 1-motives. Let \mathcal{A} be a category of abelian sheaves, and consider two bounded complexes C_1, C_2 of objects of $\mathcal{A}^{[-1, 0]}$. Let $H \in \mathcal{A}$. We have a double complex

$$\underline{\mathrm{Biext}}(C_1, C_2; H)^{p, q} := \mathrm{Biext}(C_1^p, C_2^q; H).$$

4.2.1. Definition. A *biextension of C_1 and C_2 by H* is an element of the group of cycles

$$\mathrm{Biext}(C_1, C_2; H) := Z^0(\mathbf{Tot} \underline{\mathrm{Biext}}(C_1, C_2; H)).$$

Here \mathbf{Tot} denotes the total complex associated to a double complex.

Concretely: such a biextension P is given by a collection of biextensions $P_p \in \mathrm{Biext}(C_1^p, C_2^{-p}; H)$ such that, for any p ,

$$(d_1^p \otimes 1)^* P_{p+1} = (1 \otimes d_2^{-p-1})^* P_p$$

where d_1 (*resp.* d_2) are the differentials of C_1 (*resp.* of C_2).

Now suppose that \mathcal{A} is the category of fppf sheaves, that $H = \mathbb{G}_m$ and that all the C_i^j are Deligne 1-motives. By Lemma 4.1.2, we have

$$\mathbb{E}xt^i(C_1^p, C_2^q; \mathbb{G}_m) = 0 \text{ for } i \leq 0.$$

Therefore, a spectral sequence argument yields an edge homomorphism

$$(4.9) \quad \text{Biext}(C_1, C_2; \mathbb{G}_m) \rightarrow \mathbb{E}xt^1(C_1 \overset{L}{\otimes} C_2, \mathbb{G}_m).$$

Recall that Deligne's Cartier duality [25] provides an exact functor

$$M \mapsto M^* : \mathcal{M}_1[1/p] \rightarrow \mathcal{M}_1[1/p]$$

yielding by Proposition 1.8.4 a triangulated functor

$$(4.10) \quad (\)^* : D^b(\mathcal{M}_1[1/p]) \rightarrow D^b(\mathcal{M}_1[1/p]).$$

Note that for a complex of 1-motives

$$C = (\cdots \rightarrow M^i \rightarrow M^{i+1} \rightarrow \cdots)$$

we can compute C^* by means of the complex

$$C^* = (\cdots \rightarrow (M^{i+1})^* \rightarrow (M^i)^* \rightarrow \cdots)$$

of Cartier duals here placed in degrees $\dots, -i-1, -i$, etc.

Let us now take in (4.9) $C_1 = C$, $C_2 = C^*$. For each $p \in \mathbb{Z}$, we have the Poincaré biextension $P_p \in \text{Biext}(C^p, (C^p)^*; \mathbb{G}_m)$. By Proposition 4.1.1, the $\{P_p\}$ define a class in $\text{Biext}(C, C^*; \mathbb{G}_m)$.

4.2.2. Definition. This class P_C is the *Poincaré biextension of the complex C* .

Let $C_1, C_2 \in C^b(\mathcal{M}_1)$. As in Subsection 4.1, pulling back ${}^tP_{C_1} = P_{C_1^*} \in \text{Biext}(C_1, C_1^*; \mathbb{G}_m)$ yields a map generalising (4.2):

$$(4.11) \quad \gamma_{C_1, C_2} : \text{Hom}(C_1, C_2^*) \rightarrow \text{Biext}(C_1, C_2; \mathbb{G}_m)$$

$$\varphi \mapsto (\varphi \times 1_{C_1})^*({}^tP_{C_2}).$$

which is clearly additive and natural in C_1 . We then have the following trivial extension of the functoriality in Proposition 4.1.1:

4.2.3. Proposition. γ_{C_1, C_2} is also natural in C_2 . □

4.3. A pairing with finite coefficients. In this section, we assume that k is algebraically closed.

If C is a complex of 1-motives and $n > 1$ is prime to $\text{char } k$, we define

$$C/n = \text{cone}(C \xrightarrow{n} C)$$

the mapping cone of multiplication by n . This defines a functor on $C^b(\mathcal{M}_1)$; we clearly have a natural isomorphism

$$C^*/n \simeq (C/n[-1])^* \simeq (C/n)^*[1].$$

This functor and this natural isomorphism are easily seen to pass to $D^b(\mathcal{M}_1)$. Composition of morphisms now yields a pairing (Hom groups computed in $D^b(\mathcal{M}_1)$):

$$\begin{aligned} \text{Hom}(\mathbb{Z}, C/n) \times \text{Hom}(\mathbb{Z}, C^*/n) &= \text{Hom}(\mathbb{Z}, C/n) \times \text{Hom}(\mathbb{Z}, (C/n)^*[1]) \\ &\simeq \text{Hom}(\mathbb{Z}, C/n) \times \text{Hom}(C/n, \mathbb{Z}^*[1]) \rightarrow \text{Hom}(\mathbb{Z}, \mathbb{Z}^*[1]) = k^*. \end{aligned}$$

4.3.1. Lemma. *For any $C \in C^b(\mathcal{M}_1)$, the map $C/n \xrightarrow{n} C/n$ is homotopic to 0.*

Proof. We may embed 1-motives in a category of complexes of sheaves of length 1, and then the proof is standard. \square

This lemma implies that the above pairing refines into a pairing

$$(4.12) \quad \text{Hom}(\mathbb{Z}, C/n) \times \text{Hom}(\mathbb{Z}, C^*/n) \rightarrow \mu_n.$$

4.3.2. Theorem. *This pairing is perfect.*

Proof. Convert (4.12) into a morphism

$$\text{Hom}(\mathbb{Z}, C/n) \rightarrow \text{Hom}(\text{Hom}(\mathbb{Z}, C^*/n), \mu_n).$$

This map is clearly natural in C , hence by dévissage we may check that it is an isomorphism on “generators” $C = N[i]$, $i \in \mathbb{Z}$, where N is a 1-motive. We may further reduce to $N = [\mathbb{Z} \rightarrow 0]$, $[0 \rightarrow \mathbb{G}_m]$ or $[0 \rightarrow A]$ where A is an abelian variety.

It is convenient to replace $D^b(\mathcal{M}_1)$ by $D^b({}^t\mathcal{M}_1)$ (Theorem 1.6.1), which allows us to represent N/n by

$$\begin{cases} [\mathbb{Z}/n \rightarrow 0][0] & \text{if } N = [\mathbb{Z} \rightarrow 0] \\ [\mu_n \rightarrow 0][0] & \text{if } N = [0 \rightarrow \mathbb{G}_m] \\ [{}_nA \rightarrow 0][0] & \text{if } N = [0 \rightarrow A]. \end{cases}$$

This implies that $\text{Hom}(\mathbb{Z}, N[i]/n) = 0$ if $i \neq 0$ (the best way to see this is to use the functor Tot). Suppose $i = 0$. In the cases $N = [\mathbb{Z} \rightarrow 0]$ or $[0 \rightarrow \mathbb{G}_m]$, the pairing is easily seen to be the obvious pairing $\mathbb{Z}/n \times \mu_n \rightarrow \mu_n$ or $\mu_n \times \mathbb{Z}/n \rightarrow \mu_n$, which is clearly perfect. In the case

$N = [0 \rightarrow A]$, so that $N^* = [0 \rightarrow A']$ is the dual abelian variety, we get a pairing

$${}_n A \times {}_n A' \rightarrow \mu_n$$

which is by construction the Weil pairing (see [65, IV.20] and [63, §16]). Therefore it is perfect too. \square

4.4. Comparing two Ext groups. The aim of this subsection is to prove:

4.4.1. Proposition. *Let $C_1, C_2 \in D^b(\mathcal{M}_1[1/p])$ and $C_3 \in \mathrm{DM}_{-, \text{ét}}^{\text{eff}}(k)$. Then the forgetful triangulated functors*

$$\mathrm{DM}_{-, \text{ét}}^{\text{eff}}(k) \rightarrow D^-(\mathrm{Shv}_{\text{ét}}(\mathrm{SmCor}(k))) \rightarrow D^-(\mathrm{Shv}_{\text{ét}}(\mathrm{Sm}(k)))$$

induce an isomorphism

$$\mathrm{Hom}_{\mathrm{DM}_{-, \text{ét}}^{\text{eff}}}(\mathrm{Tot} C_1 \otimes \mathrm{Tot} C_2, C_3[q]) \xrightarrow{\sim} \mathrm{Hom}_{D^-(\mathrm{Shv}_{\text{ét}})}(C_1 \overset{L}{\otimes} C_2, C_3[q])$$

for any $q \in \mathbb{Z}$.

Proof. Each of the two functors in Proposition 4.4.1 has a left adjoint. For the first one see [86, Prop. 3.2.3]; we shall denote it by RC as in *loc. cit.* The second may be constructed using [86, Remark 1 p. 202]: we shall denote it by Φ . In both cases, the construction is done for the Nisnevich topology but carries over for the étale topology as well (see also [61]). The tensor product $\overset{L}{\otimes}_{tr}$ in $D^-(\mathrm{EST})$ is defined from the formula

$$L(X) \otimes L(Y) = L(X \times Y)$$

see [86, p. 206]. For $X \in \mathrm{Sm}(k)$, let $\mathbb{Z}(X)$ be the \mathbb{Z} -free étale sheaf on the representable sheaf $Y \mapsto \mathrm{Map}_k(Y, X)$. It is clear that

$$\Phi \mathbb{Z}(X) = L(X)$$

$$\mathbb{Z}(X) \otimes \mathbb{Z}(Y) = \mathbb{Z}(X \times Y).$$

From this it follows that one has natural isomorphisms

$$\Phi(A \overset{L}{\otimes} B) = \Phi(A) \overset{L}{\otimes}_{tr} \Phi(B).$$

On the other hand, the tensor product in $\mathrm{DM}_{-, \text{ét}}^{\text{eff}}$ is defined by descent of $\overset{L}{\otimes}_{tr}$ via RC [86, p. 210]. Hence we get an adjunction isomorphism

$$\begin{aligned} \mathrm{Hom}_{D^-(\mathrm{Shv}_{\text{ét}})}(C_1 \overset{L}{\otimes} C_2, C_3[q]) &\simeq \mathrm{Hom}_{\mathrm{DM}_{-, \text{ét}}^{\text{eff}}}(RC \circ \Phi(C_1 \overset{L}{\otimes} C_2), C_3[q]) \\ &\simeq \mathrm{Hom}_{\mathrm{DM}_{-, \text{ét}}^{\text{eff}}}(RC \circ \Phi(C_1) \otimes RC \circ \Phi(C_2), C_3[q]). \end{aligned}$$

Now, since the components of C_1 and C_2 belong to $\mathrm{HI}_{\mathrm{\acute{e}t}}$, the counit maps $RC \circ \Phi(C_1) \rightarrow \mathrm{Tot}(C_1)$ and $RC \circ \Phi(C_2) \rightarrow \mathrm{Tot}(C_2)$ are isomorphisms. This concludes the proof. \square

4.5. Two Cartier dualities. Recall the internal Hom $\underline{\mathrm{Hom}}_{\mathrm{\acute{e}t}}$ from §2.5. We define

$$(4.13) \quad D_{\leq 1}^{\mathrm{\acute{e}t}}(M) := \underline{\mathrm{Hom}}_{\mathrm{\acute{e}t}}(M, \mathbb{Z}(1))$$

for any object $M \in \mathrm{DM}_{\mathrm{gm}, \mathrm{\acute{e}t}}^{\mathrm{eff}}$.

We now want to compare the duality (4.10) with the following duality on triangulated 1-motives:

4.5.1. Proposition. *The functor $D_{\leq 1}^{\mathrm{\acute{e}t}}$ restricts to a self-duality $(\)^\vee$ (anti-equivalence of categories) on $d_{\leq 1} \mathrm{DM}_{\mathrm{gm}, \mathrm{\acute{e}t}}^{\mathrm{eff}}$.*

Proof. It suffices to compute on motives of smooth projective curves $M(C)$. Then it is obvious in view of Proposition 2.5.2 c). \square

4.5.2. Definition. For $M \in d_{\leq 1} \mathrm{DM}_{\mathrm{gm}, \mathrm{\acute{e}t}}^{\mathrm{eff}}$, we say that M^\vee is the *motivic Cartier dual* of M .

Note that motivic Cartier duality exchanges Artin motives and Tate motives, *e.g.* $\mathbb{Z}^\vee = \mathbb{Z}(1)$. We are going to compare it with the Cartier duality on $D^b(\mathcal{M}_1[1/p])$ (see Proposition 1.8.4) via Theorem 2.1.2.

For two complexes of 1-motives C_1 and C_2 , by composing (4.11) and (4.9) and applying Proposition 4.2.3, we get a bifunctorial morphism

$$(4.14) \quad \mathrm{Hom}(C_1, C_2^*) \rightarrow \mathrm{Biext}(C_1, C_2; \mathbb{G}_m) \rightarrow \mathrm{Hom}(C_1 \overset{L}{\otimes} C_2, \mathbb{G}_m[-1])$$

where the right hand side is computed in the derived category of étale sheaves. This natural transformation trivially factors through $D^b(\mathcal{M}_1[1/p])$.

From Proposition 4.4.1 and [61, Thm. 4.1], taking $C_3 = \mathbb{Z}(1) \cong \mathbb{G}_m[-1]$, it follows that the map (4.14) may be reinterpreted as a natural transformation

$$\mathrm{Hom}_{D^b(\mathcal{M}_1[1/p])}(C_1, C_2^*) \rightarrow \mathrm{Hom}_{d_{\leq 1} \mathrm{DM}_{\mathrm{gm}, \mathrm{\acute{e}t}}^{\mathrm{eff}}}(\mathrm{Tot}(C_1), \mathrm{Tot}(C_2)^\vee).$$

Now we argue à la Yoneda: taking $C_1 = C$ and $C_2 = C^*$, the image of the identity yields a canonical morphism of functors:

$$\eta_C : \mathrm{Tot}(C^*) \rightarrow \mathrm{Tot}(C)^\vee.$$

4.5.3. Theorem. *The natural transformation η is an isomorphism of functors.*

Proof. It suffices to check this on 1-motives, since they are dense in the triangulated category $D^b(\mathcal{M}_1[1/p])$. Using Yoneda again and the previous discussion, it then follows from Theorem 2.1.2 and the isomorphisms (4.1) and (4.2) (the latter being proven in Proposition 4.1.1). The following commutative diagram explains this:

$$\begin{array}{ccc}
 \mathrm{Hom}(N, M^*) & \xrightarrow[\sim]{\text{Th. 2.1.2}} & \mathrm{Hom}(\mathrm{Tot}(N), \mathrm{Tot}(M^*)) \\
 \wr \downarrow (4.1)+(4.2) & & \eta_* \downarrow \\
 \mathbb{E}xt_{Sm(k)_{\text{ét}}}^1(N \overset{L}{\otimes} M, \mathbb{G}_m) & \xleftarrow[\sim]{\text{Prop. 4.4.1}} & \mathrm{Hom}(\mathrm{Tot}(N), \mathrm{Tot}(M)^\vee).
 \end{array}$$

□

Part 2. The functors LAlb and RPic

5. DEFINITION OF LAlb AND RPic

The aim of this section is to construct the closest integral approximation to a left adjoint of the full embedding Tot of Definition 2.7.1. In order to work it out, we first recollect some ideas from [84].

We shall show in Theorem 6.2.1 that the functor LAlb of Definition 5.2.1 does provide a left adjoint to Tot after we tensor Hom groups with \mathbb{Q} : this will provide a proof of Pretheorems announced in [84, Preth. 0.0.18] and [85]. See Remark 5.2.2 for an integral caveat.

5.1. Motivic Cartier duality. Recall the functor $D_{\leq 1}^{\text{ét}} : \mathrm{DM}_{\mathrm{gm}, \text{ét}}^{\text{eff}} \rightarrow \mathrm{DM}_{-, \text{ét}}^{\text{eff}}$ of (4.13). On the other hand, by Corollary D.3.3 and Theorem D.2.2, we may consider truncation on $\mathrm{DM}_{\mathrm{gm}, \text{ét}}^{\text{eff}}$ with respect to the homotopy t -structure. We have:

5.1.1. Lemma. *Let $p : X \rightarrow \mathrm{Spec} k$ be a smooth variety. Then the truncated complex $\tau_{\leq 2} D_{\leq 1}^{\text{ét}}(M_{\text{ét}}(X))$ belongs to $d_{\leq 1} \mathrm{DM}_{\mathrm{gm}, \text{ét}}^{\text{eff}}$ (here we set $M_{\text{ét}}(X) := \alpha^* M(X)$).*

Proof. The same computation as in the proof of Proposition 2.5.2 yields that the nonvanishing cohomology sheaves are $\mathcal{H}^1 = R_{\pi_0(X)/k} \mathbb{G}_m[1/p]$ and $\mathcal{H}^2 = \mathrm{Pic}_{X/k}[1/p]$. Both belong to Shv_1 (the latter by Proposition 3.4.1), hence the claim follows from Theorem 3.9.1. □

Unfortunately, $\mathcal{H}^i(D_{\leq 1}^{\text{ét}}(M_{\text{ét}}(X)))$ does not belong to $d_{\leq 1} \mathrm{DM}_{\mathrm{gm}, \text{ét}}^{\text{eff}}$ for $i > 2$ in general: indeed, it is well-known that this is a torsion sheaf of cofinite type, with nonzero divisible part in general (for $i \geq 3$ and in characteristic 0, its corank is equal to the i -th Betti number of X). It might be considered as an ind-object of $d_{\leq 0} \mathrm{DM}_{\mathrm{gm}, \text{ét}}^{\text{eff}}$, but this would

take us too far. To get around this problem, we shall restrict to the standard category of geometric triangulated motives of [86], $\mathrm{DM}_{\mathrm{gm}}^{\mathrm{eff}}$.

Let us denote by $D_{\leq 1}^{\mathrm{Nis}}$ the same functor as $D_{\leq 1}^{\mathrm{\acute{e}t}}$ in the category $\mathrm{DM}_{-}^{\mathrm{eff}}$, defined with the Nisnevich topology. Let as before $\alpha^* : \mathrm{DM}_{-}^{\mathrm{eff}} \rightarrow \mathrm{DM}_{-, \mathrm{\acute{e}t}}^{\mathrm{eff}}$ denote the “change of topology” functor.

5.1.2. Lemma. *a) For any smooth X with motive $M(X) \in \mathrm{DM}_{\mathrm{gm}}^{\mathrm{eff}}$, we have*

$$\alpha^* D_{\leq 1}^{\mathrm{Nis}} M(X) \xrightarrow{\sim} \tau_{\leq 2} D_{\leq 1}^{\mathrm{\acute{e}t}} \alpha^* M(X).$$

b) The functor $\alpha^ D_{\leq 1}^{\mathrm{Nis}}$ induces a triangulated functor*

$$\alpha^* D_{\leq 1}^{\mathrm{Nis}} : \mathrm{DM}_{\mathrm{gm}}^{\mathrm{eff}} \rightarrow d_{\leq 1} \mathrm{DM}_{\mathrm{gm}, \mathrm{\acute{e}t}}^{\mathrm{eff}}.$$

Proof. a) This is the weight 1 case of the Beilinson-Lichtenbaum conjecture (here equivalent to Hilbert’s theorem 90.) b) follows from a) and Lemma 5.1.1. \square

5.1.3. Definition. We denote by $d_{\leq 1} : \mathrm{DM}_{\mathrm{gm}}^{\mathrm{eff}} \rightarrow d_{\leq 1} \mathrm{DM}_{\mathrm{gm}, \mathrm{\acute{e}t}}^{\mathrm{eff}}$ the composite functor $D_{\leq 1}^{\mathrm{\acute{e}t}} \circ \alpha^* \circ D_{\leq 1}^{\mathrm{Nis}}$.

Thus, for $M \in \mathrm{DM}_{\mathrm{gm}}^{\mathrm{eff}}$, we have

$$(5.1) \quad d_{\leq 1}(M) = \underline{\mathrm{Hom}}_{\mathrm{\acute{e}t}}(\alpha^* \underline{\mathrm{Hom}}_{\mathrm{Nis}}(M, \mathbb{Z}(1)), \mathbb{Z}(1)).$$

The evaluation map $M \otimes \underline{\mathrm{Hom}}_{\mathrm{Nis}}(M, \mathbb{Z}(1)) \rightarrow \mathbb{Z}(1)$ then yields a canonical map

$$(5.2) \quad a_M : \alpha^* M \rightarrow d_{\leq 1}(M)$$

for any object $M \in \mathrm{DM}_{\mathrm{gm}}^{\mathrm{eff}}$. We call a_M the *motivic Albanese map* associated to M for reasons that will appear later.

5.1.4. Proposition. *The restriction of (5.2) to $d_{\leq 1} \mathrm{DM}_{\mathrm{gm}}^{\mathrm{eff}}$ is an isomorphism of functors. In particular, we have an equality*

$$\alpha^* D_{\leq 1}(\mathrm{DM}_{\mathrm{gm}}^{\mathrm{eff}}) = \alpha^* d_{\leq 1} \mathrm{DM}_{\mathrm{gm}}^{\mathrm{eff}}.$$

Proof. For the first claim, we reduce to the case $M = M(C)$ where C is a smooth proper curve. The argument is then exactly the same as in Proposition 4.5.1, using (2.2). The other claim is then clear. \square

5.2. Motivic Albanese.

5.2.1. Definition. The *motivic Albanese functor*

$$\mathrm{LAlb} : \mathrm{DM}_{\mathrm{gm}}^{\mathrm{eff}} \rightarrow D^b(\mathcal{M}_1[1/p])$$

is the composition of $d_{\leq 1}$ with a quasi-inverse to the equivalence of categories of Theorem 2.1.2.

5.2.2. Remark. With this definition we get the following form of (5.2)

$$a_M : \alpha^* M \rightarrow \text{Tot LAlb}(M)$$

By Theorem 2.1.2, we then have the following relationship between LAlb and the functor Tot of Definition 2.7.1: for $M \in \text{DM}_{\text{gm}}^{\text{eff}}$ and $N \in D^b(\mathcal{M}_1[1/p])$, the map a_M of (5.2) induces a map

$$(5.3) \quad \text{Hom}(\text{LAlb } M, N) \rightarrow \text{Hom}(\alpha^* M, \text{Tot}(N)).$$

In Section 6, we prove that this map is an isomorphism rationally, showing that LAlb yields a left adjoint of Tot after Hom groups have been tensored with \mathbb{Q} . However, it is not so in general⁹: *e.g.* take $M = \mathbb{Z}(2)$, $N = \mathbb{Z}/n$ (n prime to p). Then $\text{LAlb } M = 0$ because $\underline{\text{Hom}}_{\text{Nis}}(\mathbb{Z}(2), \mathbb{Z}(1)) = 0$, but

$$\text{Hom}(\alpha^* M, \text{Tot}(N)) = H_{\text{ét}}^0(k, \mathbb{Z}/n(-2))$$

which is in general nonzero.

The same example shows that Tot does not have a left adjoint. Indeed, suppose that such a left adjoint exists, and let us denote it by $\text{LAlb}^{\text{ét}}$. For simplicity, suppose k algebraically closed. Let $n \geq 2$. For any $m > 0$, the exact triangle in $\text{DM}_{\text{gm}, \text{ét}}^{\text{eff}}$

$$\mathbb{Z}(n) \xrightarrow{m} \mathbb{Z}(n) \rightarrow \mathbb{Z}/m(n) \xrightarrow{+1}$$

must yield an exact triangle

$$\text{LAlb}^{\text{ét}} \mathbb{Z}(n) \xrightarrow{m} \text{LAlb}^{\text{ét}} \mathbb{Z}(n) \rightarrow \text{LAlb}^{\text{ét}} \mathbb{Z}/m(n) \xrightarrow{+1}.$$

Since Tot is an equivalence on torsion objects, so must be $\text{LAlb}^{\text{ét}}$. Since k is algebraically closed, $\mathbb{Z}/m(n) \simeq \mu_m^{\otimes n}$ is constant, hence we must have $\text{LAlb}^{\text{ét}} \mathbb{Z}/m(n) \simeq [\mathbb{Z}/m \rightarrow 0]$. Hence, multiplication by m must be bijective on the 1-motives $H^q(\text{LAlb}^{\text{ét}}(\mathbb{Z}(n)))$ for all $q \neq 0, 1$, which forces these 1-motives to vanish. For $q = 0, 1$ we must have exact sequences

$$\begin{aligned} 0 \rightarrow H^0(\text{LAlb}^{\text{ét}}(\mathbb{Z}(n))) &\xrightarrow{m} H^0(\text{LAlb}^{\text{ét}}(\mathbb{Z}(n))) \rightarrow [\mathbb{Z}/m \rightarrow 0] \\ &\rightarrow H^1(\text{LAlb}^{\text{ét}}(\mathbb{Z}(n))) \xrightarrow{m} H^1(\text{LAlb}^{\text{ét}}(\mathbb{Z}(n))) \rightarrow 0 \end{aligned}$$

which force either $H^0 = [\mathbb{Z} \rightarrow 0]$, $H^1 = 0$ or $H^0 = 0$, $H^1 = [0 \rightarrow \mathbb{G}_m]$. But both cases are impossible as one easily sees by computing

$$\begin{aligned} \text{Hom}(M(\mathbb{P}^n), \text{Tot}([\mathbb{Z} \rightarrow 0])[2n+1]) &= H_{\text{ét}}^{2n+1}(\mathbb{P}^n, \mathbb{Z})[1/p] \\ &\simeq H_{\text{ét}}^{2n}(\mathbb{P}^n, (\mathbb{Q}/\mathbb{Z})') \simeq (\mathbb{Q}/\mathbb{Z})' \end{aligned}$$

via the trace map, where $(\mathbb{Q}/\mathbb{Z})' = \bigoplus_{\ell \neq p} \mathbb{Q}_{\ell}/\mathbb{Z}_{\ell}$.

⁹Contrary to what was claimed in a preliminary version of this paper.

Presumably, $\mathrm{LAlb}^{\acute{\mathrm{e}}\mathrm{t}}$ does exist with values in a suitable pro-category containing $D^b(\mathcal{M}_1[1/p])$, and sends $\mathbb{Z}(n)$ to the complete Tate module of $\mathbb{Z}(n)$ for $n \geq 2$. Note that, by 8.1.c below, $\mathrm{LAlb}(\mathbb{Z}(n)) = 0$ for $n \geq 2$, so that the natural transformation $\mathrm{LAlb}^{\acute{\mathrm{e}}\mathrm{t}}(\alpha^*M) \rightarrow \mathrm{LAlb}(M)$ will not be an isomorphism of functors in general.

5.3. Motivic Pic.

5.3.1. Definition. The *motivic Picard functor* (a contravariant functor) is the functor

$$\mathrm{RPic} : \mathrm{DM}_{\mathrm{gm}}^{\mathrm{eff}} \rightarrow D^b(\mathcal{M}_1[1/p])$$

given by $\mathrm{Tot}^{-1} \alpha^* D_{\leq 1}^{\mathrm{Nis}}$ (cf. Definition 5.2.1).

For $M \in \mathrm{DM}_{\mathrm{gm}}^{\mathrm{eff}}$ we then have the following tautology

$$(\mathrm{Tot} \mathrm{RPic}(M))^{\vee} = \mathrm{Tot} \mathrm{LAlb}(M).$$

Actually, from Theorem 4.5.3 we deduce:

5.3.2. Corollary. For $M \in \mathrm{DM}_{\mathrm{gm}}^{\mathrm{eff}}$ we have

$$\mathrm{RPic}(M)^* = \mathrm{LAlb}(M). \quad \square$$

Therefore we get ${}^t H^i(\mathrm{RPic}(M)) = ({}^t H_i(\mathrm{LAlb}(M)))^*$.

6. THE ADJUNCTION $\mathrm{LAlb} - \mathrm{Tot}$ WITH RATIONAL COEFFICIENTS

Throughout this section, we use the notations $\otimes \mathbb{Q}$ and $\boxtimes \mathbb{Q}$ from Definition 1.1.3.

6.1. Rational coefficients revisited. Let $\mathrm{DM}_{-}^{\mathrm{eff}}(k; \mathbb{Q})$ and $\mathrm{DM}_{-, \acute{\mathrm{e}}\mathrm{t}}^{\mathrm{eff}}(k; \mathbb{Q})$ denote the full subcategories of $\mathrm{DM}_{-}^{\mathrm{eff}}(k)$ and $\mathrm{DM}_{-, \acute{\mathrm{e}}\mathrm{t}}^{\mathrm{eff}}(k)$ formed of those complexes whose cohomology sheaves are uniquely divisible. Recall that by [86, Prop. 3.3.2], the change of topology functor

$$\alpha^* : \mathrm{DM}_{-}^{\mathrm{eff}}(k) \rightarrow \mathrm{DM}_{-, \acute{\mathrm{e}}\mathrm{t}}^{\mathrm{eff}}(k)$$

induces an equivalence of categories

$$\mathrm{DM}_{-}^{\mathrm{eff}}(k; \mathbb{Q}) \xrightarrow{\sim} \mathrm{DM}_{-, \acute{\mathrm{e}}\mathrm{t}}^{\mathrm{eff}}(k; \mathbb{Q}).$$

Beware that in *loc. cit.*, these two categories are respectively denoted by $\mathrm{DM}_{-}^{\mathrm{eff}}(k) \otimes \mathbb{Q}$ and $\mathrm{DM}_{-, \acute{\mathrm{e}}\mathrm{t}}^{\mathrm{eff}}(k) \otimes \mathbb{Q}$, while this notation is used here according to Definition 1.1.3. The composite functors (with our notation)

$$\begin{aligned} \mathrm{DM}_{-}^{\mathrm{eff}}(k; \mathbb{Q}) &\rightarrow \mathrm{DM}_{-}^{\mathrm{eff}}(k) \rightarrow \mathrm{DM}_{-}^{\mathrm{eff}}(k) \otimes \mathbb{Q} \\ \mathrm{DM}_{-, \acute{\mathrm{e}}\mathrm{t}}^{\mathrm{eff}}(k; \mathbb{Q}) &\rightarrow \mathrm{DM}_{-, \acute{\mathrm{e}}\mathrm{t}}^{\mathrm{eff}}(k) \rightarrow \mathrm{DM}_{-, \acute{\mathrm{e}}\mathrm{t}}^{\mathrm{eff}}(k) \otimes \mathbb{Q} \end{aligned}$$

are fully faithful but not essentially surjective. The functor $\alpha^* \otimes \mathbb{Q}$ is not essentially surjective, nor (a priori) fully faithful. Nevertheless, these two composite functors have a left adjoint/left inverse $C \mapsto C \otimes \mathbb{Q}$, and

6.1.1. Proposition. *a) The compositions*

$$\begin{aligned} \mathrm{DM}_{\mathrm{gm}}^{\mathrm{eff}}(k) \otimes \mathbb{Q} &\rightarrow \mathrm{DM}_{-}^{\mathrm{eff}}(k) \otimes \mathbb{Q} \xrightarrow{\otimes \mathbb{Q}} \mathrm{DM}_{-}^{\mathrm{eff}}(k; \mathbb{Q}) \\ \mathrm{DM}_{\mathrm{gm}, \mathrm{\acute{e}t}}^{\mathrm{eff}}(k) \otimes \mathbb{Q} &\rightarrow \mathrm{DM}_{-, \mathrm{\acute{e}t}}^{\mathrm{eff}}(k) \otimes \mathbb{Q} \xrightarrow{\otimes \mathbb{Q}} \mathrm{DM}_{-, \mathrm{\acute{e}t}}^{\mathrm{eff}}(k; \mathbb{Q}) \end{aligned}$$

are fully faithful.

b) Via these full embeddings, the functor α^ induces equivalences of categories*

$$\begin{aligned} \mathrm{DM}_{\mathrm{gm}}^{\mathrm{eff}}(k) \boxtimes \mathbb{Q} &\xrightarrow{\sim} \mathrm{DM}_{\mathrm{gm}, \mathrm{\acute{e}t}}^{\mathrm{eff}}(k) \boxtimes \mathbb{Q} \\ d_{\leq 1} \mathrm{DM}_{\mathrm{gm}}^{\mathrm{eff}}(k) \boxtimes \mathbb{Q} &\xrightarrow{\sim} d_{\leq 1} \mathrm{DM}_{\mathrm{gm}, \mathrm{\acute{e}t}}^{\mathrm{eff}}(k) \boxtimes \mathbb{Q}. \end{aligned}$$

Here $d_{\leq 1} \mathrm{DM}_{\mathrm{gm}}^{\mathrm{eff}}(k)$ is the thick subcategory of $\mathrm{DM}_{\mathrm{gm}}^{\mathrm{eff}}(k)$ generated by motives of smooth curves.

Proof. a) We shall give it for the first composition (for the second one it is similar). Let $M, N \in \mathrm{DM}_{\mathrm{gm}}^{\mathrm{eff}}(k)$: we have to prove that the obvious map

$$\mathrm{Hom}(M, N) \otimes \mathbb{Q} \rightarrow \mathrm{Hom}(M \otimes \mathbb{Q}, N \otimes \mathbb{Q})$$

is an isomorphism. We shall actually prove this isomorphism for any $M \in \mathrm{DM}_{\mathrm{gm}}^{\mathrm{eff}}(k)$ and any $N \in \mathrm{DM}_{-}^{\mathrm{eff}}(k)$. By adjunction, the right hand side coincides with $\mathrm{Hom}(M, N \otimes \mathbb{Q})$ computed in $\mathrm{DM}_{-}^{\mathrm{eff}}(k)$. We may reduce to $M = M(X)$ for X smooth. By [86, Prop. 3.2.8], we are left to see that the map

$$H_{\mathrm{Nis}}^q(X, N) \otimes \mathbb{Q} \rightarrow H_{\mathrm{Nis}}^q(X, N \otimes \mathbb{Q})$$

is an isomorphism for any $q \in \mathbb{Z}$. By hypercohomology spectral sequences, we reduce to the case where N is a sheaf concentrated in degree 0; then the assertion follows from the fact that Nisnevich cohomology commutes with filtering direct limits of sheaves.

b) It is clear that the two compositions commute with α^* , which sends $\mathrm{DM}_{\mathrm{gm}}^{\mathrm{eff}}(k) \otimes \mathbb{Q}$ into $\mathrm{DM}_{\mathrm{gm}, \mathrm{\acute{e}t}}^{\mathrm{eff}}(k) \otimes \mathbb{Q}$. By a) and [86, Prop. 3.3.2], this functor is fully faithful, and the induced functor on the \boxtimes categories remains so and is essentially surjective by definition of the two categories. Similarly for the $d_{\leq 1}$ categories. \square

6.1.2. Remarks. 1) In fact, $d_{\leq 1} \mathrm{DM}_{\mathrm{gm}, \mathrm{\acute{e}t}}^{\mathrm{eff}}(k) \otimes \mathbb{Q} = d_{\leq 1} \mathrm{DM}_{\mathrm{gm}, \mathrm{\acute{e}t}}^{\mathrm{eff}}(k) \boxtimes \mathbb{Q}$ thanks to Corollary 1.6.2 and Theorem 2.1.2. We don't know whether the same is true for the other categories.

2) See [73, A.2.2] for a different, more general approach to Proposition 6.1.1.

6.1.3. Definition. With rational coefficients, we identify $\mathrm{DM}_{\mathrm{gm}}^{\mathrm{eff}}(k) \boxtimes \mathbb{Q}$ with $\mathrm{DM}_{\mathrm{gm}, \mathrm{\acute{e}t}}^{\mathrm{eff}}(k) \boxtimes \mathbb{Q}$ via α^* (using Proposition 6.1.1 b)) and define

$$\begin{aligned} D_{\leq 1} &= D_{\leq 1}^{\mathrm{Nis}} = D_{\leq 1}^{\mathrm{\acute{e}t}} \\ d_{\leq 1} &= D_{\leq 1}^2 \\ \mathrm{LAlb}^{\mathbb{Q}} &= \mathrm{Tot}^{-1} \circ d_{\leq 1}. \end{aligned}$$

6.1.4. Remark. This definition is compatible with the formula $d_{\leq 1} = D_{\leq 1}^{\mathrm{\acute{e}t}} \circ \alpha^* \circ D_{\leq 1}^{\mathrm{Nis}}$ of Definition 5.1.3.

6.2. The functor $\mathrm{LAlb}^{\mathbb{Q}}$. We now get the announced adjunction by taking (5.1) with rational coefficients, thanks to Corollary 1.6.2 and Proposition 6.1.1.

6.2.1. Theorem. *Let $M \in \mathrm{DM}_{\mathrm{gm}}^{\mathrm{eff}}(k) \boxtimes \mathbb{Q}$. Then the map a_M from (5.2) induces an isomorphism*

$$\mathrm{Hom}(d_{\leq 1}M, M') \xrightarrow{\sim} \mathrm{Hom}(M, M')$$

for any $M' \in d_{\leq 1} \mathrm{DM}_{\mathrm{gm}}^{\mathrm{eff}}(k) \boxtimes \mathbb{Q}$. Equivalently, (5.3) is an isomorphism with rational coefficients.

Proof. By Proposition 4.5.1, M' can be written as $N^{\vee} = D_{\leq 1}(N)$ for some $N \in d_{\leq 1} \mathrm{DM}_{\mathrm{gm}}^{\mathrm{eff}}(k) \boxtimes \mathbb{Q}$. We have the following commutative diagram

$$\begin{array}{ccccc} \mathrm{Hom}(M, D_{\leq 1}(N)) & = & \mathrm{Hom}(M \otimes N, \mathbb{Z}(1)) & = & \mathrm{Hom}(N, D_{\leq 1}(M)) \\ a_M^* \uparrow & & (a_M \otimes 1_N)^* \uparrow & & D_{\leq 1}(a_M)_* \uparrow \\ \mathrm{Hom}(d_{\leq 1}M, D_{\leq 1}(N)) & = & \mathrm{Hom}(d_{\leq 1}M \otimes N, \mathbb{Z}(1)) & = & \mathrm{Hom}(N, D_{\leq 1}(d_{\leq 1}M)). \end{array}$$

But $D_{\leq 1}(a_M) \circ a_{D_{\leq 1}M} = 1_{D_{\leq 1}M}$ [74, p. 56, (3.2.3.9)]¹⁰ and $a_{D_{\leq 1}M}$ is an isomorphism by Proposition 5.1.4, which proves the claim. \square

6.2.2. Corollary. *The functor $d_{\leq 1}$ of (5.1) induces a left adjoint to the embedding $d_{\leq 1} \mathrm{DM}_{\mathrm{gm}}^{\mathrm{eff}}(k) \boxtimes \mathbb{Q} \hookrightarrow \mathrm{DM}_{\mathrm{gm}}^{\mathrm{eff}}(k) \boxtimes \mathbb{Q}$. The Voevodsky-Orgogozo full embedding $\mathrm{Tot} : D^b(\mathcal{M}_1 \otimes \mathbb{Q}) \rightarrow \mathrm{DM}_{\mathrm{gm}}^{\mathrm{eff}} \boxtimes \mathbb{Q}$ has a left adjoint/left inverse $\mathrm{LAlb}^{\mathbb{Q}}$. \square*

7. A TENSOR STRUCTURE AND AN INTERNAL HOM ON $D^b(\mathcal{M}_1 \otimes \mathbb{Q})$

In this section, coefficients are tensored with \mathbb{Q} and we use the functor $\mathrm{LAlb}^{\mathbb{Q}}$ of Corollary 6.2.2.

¹⁰Note that this proof carries over in our case.

7.1. Tensor structure.

7.1.1. Lemma. *Let G_1, G_2 be two semi-abelian varieties. Then, we have in $\mathrm{DM}_{\mathrm{gm}, \mathrm{\acute{e}t}}^{\mathrm{eff}} \boxtimes \mathbb{Q}$:*

$$\mathcal{H}^q(D_{\leq 1}(\underline{G}_1[-1] \otimes \underline{G}_2[-1])) = \begin{cases} \mathrm{Biext}(G_1, G_2; \mathbb{G}_m) & \text{if } q = 0 \\ 0 & \text{else.} \end{cases}$$

Proof. By Gersten's principle (Proposition 2.4.1), it is enough to show that the isomorphisms are valid over function fields K of smooth k -varieties and that \mathcal{H}^0 comes from the small étale site of $\mathrm{Spec} k$. Since we work up to torsion, we may even replace K by its perfect closure. Thus, without loss of generality, we may assume $K = k$ and we have to show the lemma for sections over k .

For $q \leq 0$, we use Proposition 4.4.1: for $q < 0$ this follows from Lemma 4.1.2, while for $q = 0$ it follows from the isomorphisms (4.1) and (4.2) (see Proposition 4.1.1), which show that $\mathrm{Biext}(G_1, G_2; \mathbb{G}_m)$ is rigid.

For $q > 0$, we use the formula

$$D_{\leq 1}(\underline{G}_1[-1] \otimes \underline{G}_2[-1]) \simeq \underline{\mathrm{Hom}}(\underline{G}_1[-1], \mathrm{Tot}([0 \rightarrow G_2]^*))$$

coming from Theorem 4.5.3. Writing $[0 \rightarrow G_2]^* = [L_2 \rightarrow A_2]$ with L_2 a lattice and A_2 an abelian variety, we are left to show that

$$\begin{aligned} \mathrm{Hom}_{\mathrm{DM}_{\mathrm{gm}, \mathrm{\acute{e}t}}^{\mathrm{eff}} \boxtimes \mathbb{Q}}(\underline{G}_1, L_2[q+1]) &= 0 \text{ for } q > 0 \\ \mathrm{Hom}_{\mathrm{DM}_{\mathrm{gm}, \mathrm{\acute{e}t}}^{\mathrm{eff}} \boxtimes \mathbb{Q}}(\underline{G}_1, \underline{A}_2[q]) &= 0 \text{ for } q > 0. \end{aligned}$$

For this, we may reduce to the case where G_1 is either an abelian variety or \mathbb{G}_m . If $G_1 = \mathbb{G}_m$, \underline{G}_1 is a direct summand of $M(\mathbb{P}^1)[-1]$ and the result follows. If G_1 is an abelian variety, it is isogenous to a direct summand of $J(C)$ for C a smooth projective geometrically irreducible curve. Then \underline{G}_1 is a direct summand of $M(C)$, and the result follows again since L_2 and \underline{A}_2 define locally constant (flasque) sheaves for the Zariski topology. \square

7.1.2. Proposition. *a) The functor $\mathrm{LAlb}^{\mathbb{Q}} : \mathrm{DM}_{\mathrm{gm}}^{\mathrm{eff}} \boxtimes \mathbb{Q} \rightarrow D^b(\mathcal{M}_1 \otimes \mathbb{Q})$ is a localisation functor; it carries the tensor structure \otimes of $\mathrm{DM}_{\mathrm{gm}}^{\mathrm{eff}} \boxtimes \mathbb{Q}$ to a tensor structure \otimes_1 on $D^b(\mathcal{M}_1 \otimes \mathbb{Q})$.*

b) For $(M, N) \in \mathrm{DM}_{\mathrm{gm}}^{\mathrm{eff}} \boxtimes \mathbb{Q} \times D^b(\mathcal{M}_1 \otimes \mathbb{Q})$, we have

$$\mathrm{LAlb}^{\mathbb{Q}}(M \otimes \mathrm{Tot}(N)) \simeq \mathrm{LAlb}^{\mathbb{Q}}(M) \otimes_1 N.$$

c) We have

$$[\mathbb{Z} \rightarrow 0] \otimes_1 C = C$$

for any $C \in D^b(\mathcal{M}_1 \otimes \mathbb{Q})$;

$$N_1 \otimes_1 N_2 = [L \rightarrow G]$$

for two Deligne 1-motives $N_1 = [L_1 \rightarrow G_1]$, $N_2 = [L_2 \rightarrow G_2]$, where

$$L = L_1 \otimes L_2;$$

there is an extension

$$0 \rightarrow \mathrm{Biext}(G_1, G_2; \mathbb{G}_m)^* \rightarrow G \rightarrow L_1 \otimes G_2 \oplus L_2 \otimes G_1 \rightarrow 0.$$

d) The tensor product \otimes_1 is exact with respect to the motivic t -structure and respects the weight filtration. Moreover, it is right exact with respect to the homotopy t -structure.

e) For two 1-motives N_1, N_2 and a semi-abelian variety G , we have

$$\mathrm{Hom}(N_1 \otimes_1 N_2, [0 \rightarrow G]) \simeq \mathrm{Biext}(N_1, N_2; G) \otimes \mathbb{Q}.$$

Proof. a) The first statement is clear since LAlb is left adjoint to the fully faithful functor Tot . For the second, it suffices to see that if $\mathrm{LAlb}^{\mathbb{Q}}(M) = 0$ then $\mathrm{LAlb}^{\mathbb{Q}}(M \otimes N) = 0$ for any $N \in \mathrm{DM}_{\mathrm{gm}}^{\mathrm{eff}} \otimes \mathbb{Q}$. We may check this after applying Tot . Note that, by Proposition 4.5.1 and Remark 6.1.4 3), $\mathrm{Tot} \mathrm{LAlb}^{\mathbb{Q}}(M) = d_{\leq 1}(M) = 0$ is equivalent to $D_{\leq 1}(M) = 0$. We have:

$$D_{\leq 1}(M \otimes N) = \underline{\mathrm{Hom}}(M \otimes N, \mathbb{Z}(1)) = \underline{\mathrm{Hom}}(N, \underline{\mathrm{Hom}}(M, \mathbb{Z}(1))) = 0.$$

b) Let $M' = \mathrm{fibre}(\mathrm{Tot} \mathrm{LAlb}^{\mathbb{Q}}(M) \rightarrow M)$: then $\mathrm{LAlb}^{\mathbb{Q}}(M') = 0$. By definition of \otimes_1 we then have

$$\begin{aligned} \mathrm{LAlb}^{\mathbb{Q}}(M) \otimes_1 N &= \mathrm{LAlb}^{\mathbb{Q}}(\mathrm{Tot} \mathrm{LAlb}^{\mathbb{Q}}(M) \otimes \mathrm{Tot}(N)) \\ &\xrightarrow{\sim} \mathrm{LAlb}^{\mathbb{Q}}(M \otimes \mathrm{Tot}(N)). \end{aligned}$$

c) The first formula is obvious. For the second, we have an exact triangle

$$\begin{aligned} G_1[-1] \otimes G_2[-1] &\rightarrow \mathrm{Tot}(N_1) \otimes \mathrm{Tot}(N_2) \\ &\rightarrow \mathrm{Tot}([L_1 \otimes L_2 \rightarrow L_1 \otimes G_2 \oplus L_2 \otimes G_1]) \xrightarrow{+1} \end{aligned}$$

hence an exact triangle

$$\begin{aligned} &\underline{\mathrm{Hom}}(\mathrm{Tot}([L_1 \otimes L_2 \rightarrow L_1 \otimes G_2 \oplus L_2 \otimes G_1], \mathbb{Z}(1)) \\ &\rightarrow \underline{\mathrm{Hom}}(\mathrm{Tot}(N_1) \otimes \mathrm{Tot}(N_2), \mathbb{Z}(1)) \rightarrow \underline{\mathrm{Hom}}(G_1[-1] \otimes G_2[-1], \mathbb{G}_m) \xrightarrow{+1} \end{aligned}$$

By Lemma 7.1.1, the last term is $\mathrm{Biext}(G_1, G_2; \mathbb{G}_m)$, hence the claim.

d) Exactness and compatibility with weights follow from the second formula of b); right exactness for the homotopy t -structure holds because it holds on $\mathrm{DM}_{-}^{\mathrm{eff}} \otimes \mathbb{Q}$.

e) We have:

$$\begin{aligned} \mathrm{Hom}_{\mathcal{M}_1 \otimes \mathbb{Q}}(N_1 \otimes_1 N_2, [0 \rightarrow G]) &= \mathrm{Hom}_{d_{\leq 1} \mathrm{DM} \otimes \mathbb{Q}}(\mathrm{Tot}(N_1 \otimes_1 N_2), G[-1]) \\ &= \mathrm{Hom}_{\mathrm{DM} \otimes \mathbb{Q}}(\mathrm{Tot}(N_1) \otimes \mathrm{Tot}(N_2), G[-1]) = \mathrm{Biext}(N_1, N_2; G) \otimes \mathbb{Q} \end{aligned}$$

by Proposition 4.4.1 and formula (4.1). \square

7.1.3. Remarks. 1) It would be interesting to try and define \otimes_1 a priori, with integral coefficients, and to see whether it is compatible with the tensor product of $\mathrm{DM}_{\mathrm{gm}}^{\mathrm{eff}}$ via the integral LAlb.

2) It is likely that Proposition 7.1.2 e) generalises to an isomorphism

$$\mathrm{Hom}_{\mathcal{M}_1 \otimes \mathbb{Q}}(N_1 \otimes_1 N_2, N) = \mathrm{Biext}(N_1, N_2; N) \otimes \mathbb{Q}$$

for three 1-motives N_1, N_2, N , where the right hand side is the biextension group introduced by Cristiana Bertolin [12], but we have not tried to check it¹¹. This puts in perspective her desire to interpret these groups as Hom groups in the (future) tannakian category generated by 1-motives.

More precisely, one expects that $\mathrm{DM}_{\mathrm{gm}} \boxtimes \mathbb{Q}$ carries a motivic t -structure whose heart \mathcal{MM} would be the searched-for abelian category of mixed motives. Then $\mathcal{M}_1 \otimes \mathbb{Q}$ would be a full subcategory of \mathcal{MM} and we might consider the thick tensor subcategory $\mathcal{M}_1^{\otimes} \subseteq \mathcal{MM}$ generated by $\mathcal{M}_1 \otimes \mathbb{Q}$ and the Tate motive (inverse to the Lefschetz motive): this is the putative category Bertolin has in mind.

Since the existence of the abelian category of mixed Tate motives (to be contained in \mathcal{M}_1^{\otimes} !) depends on the truth of the Beilinson-Soulé conjecture, this basic obstruction appears here too.

Extrapolating from Corollary 6.2.2 and Proposition 7.1.2, it seems that the embedding $\mathcal{M}_1 \otimes \mathbb{Q} \hookrightarrow \mathcal{MM}^{\mathrm{eff}}$ (where $\mathcal{MM}^{\mathrm{eff}}$ is to be the intersection of \mathcal{MM} with $\mathrm{DM}_{\mathrm{gm}}^{\mathrm{eff}} \boxtimes \mathbb{Q}$) is destined to have a left adjoint/left inverse $\mathrm{Alb}^{\mathbb{Q}} = H_0 \circ \mathrm{LAlb}_{|\mathcal{MM}^{\mathrm{eff}}|^{\mathbb{Q}}}$, which would carry the tensor product of $\mathcal{MM}^{\mathrm{eff}}$ to \otimes_1 . Restricting $\mathrm{Alb}^{\mathbb{Q}}$ to $\mathcal{M}_1^{\otimes} \cap \mathcal{MM}^{\mathrm{eff}}$ would provide the link between Bertolin's ideas and Proposition 7.1.2 e).

7.2. Internal Hom.

7.2.1. Proposition. a) *The formula*

$$\underline{\mathrm{Hom}}(C_1, C_2) = (C_1 \otimes_1 C_2^*)^*$$

defines an internal Hom on $D^b(\mathcal{M}_1 \otimes \mathbb{Q})$, right adjoint to the tensor product of Proposition 7.1.2.

b) *This internal Hom is exact for the motivic t -structure.*

¹¹This was since proven in [13].

Proof. a) We have to get a natural isomorphism

$$\mathrm{Hom}(C_1 \otimes_1 C_2, C_3) \simeq \mathrm{Hom}(C_1, \underline{\mathrm{Hom}}(C_2, C_3))$$

for three objects $C_1, C_2, C_3 \in D^b(\mathcal{M}_1 \otimes \mathbb{Q})$.

Let us still write \otimes_1 for the tensor product in $d_{\leq 1} \mathrm{DM}_{\mathrm{gm}}^{\mathrm{eff}} \otimes \mathbb{Q}$ corresponding to the tensor product in $D^b(\mathcal{M}_1 \otimes \mathbb{Q})$. By definition, we have for $M_1, M_2, M \in d_{\leq 1} \mathrm{DM}$

$$\begin{aligned} \mathrm{Hom}_{d_{\leq 1} \mathrm{DM}}(M_1 \otimes_1 M_2, M) &\simeq \mathrm{Hom}_{\mathrm{DM}}(M_1 \otimes M_2, M) \\ &= \mathrm{Hom}_{\mathrm{DM}}(M_1, \underline{\mathrm{Hom}}(M_2, M)). \end{aligned}$$

In view of Theorem 4.5.3, we are left to show that $\underline{\mathrm{Hom}}(M_2, M) \simeq (M_2 \otimes_1 M^\vee)^\vee$. By duality, we may replace M by M^\vee . Then:

$$\begin{aligned} \underline{\mathrm{Hom}}(M_2, M^\vee) &= \underline{\mathrm{Hom}}(M_2, \underline{\mathrm{Hom}}(M, \mathbb{Z}(1))) \\ &\simeq \underline{\mathrm{Hom}}(M_2 \otimes M, \mathbb{Z}(1)) \simeq \underline{\mathrm{Hom}}(M_2 \otimes_1 M, \mathbb{Z}(1)) = (M_2 \otimes_1 M)^\vee \end{aligned}$$

where the second isomorphism follows from Proposition 7.1.2 b).

b) This follows from the formula of a), Proposition 7.1.2 d) and Proposition 1.8.4 c). \square

Part 3. Some computations

8. THE ALBANESE COMPLEXES AND THEIR BASIC PROPERTIES

We introduce homological and Borel-Moore Albanese complexes of an algebraic variety providing a computation of their 1-motivic homology.

We also consider a slightly more sophisticated cohomological Albanese complex $\mathrm{LAlb}^*(X)$ which is only contravariantly functorial for maps between schemes of the same dimension. All these complexes coincide for smooth proper schemes.

8.1. The homological Albanese complex. Let $p : X \rightarrow \mathrm{Spec} k$ be a smooth variety. Recall that X has a motive $M(X) \in \mathrm{DM}_{\mathrm{gm}}^{\mathrm{eff}}$ [86]: M is a covariant functor from the category $\mathrm{Sm}(k)$ of smooth k -schemes of finite type to $\mathrm{DM}_{\mathrm{gm}}^{\mathrm{eff}}$. The image of $M(X)$ via the full embedding $\mathrm{DM}_{\mathrm{gm}}^{\mathrm{eff}} \rightarrow \mathrm{DM}_-^{\mathrm{eff}}$ is given by the Suslin complex C_* of the representable Nisnevich sheaf with transfers $L(X)$ associated to X .

For X an arbitrary k -scheme of finite type, the formula $M(X) = C_*(L(X))$ still defines an object of $\mathrm{DM}_-^{\mathrm{eff}}$; if $\mathrm{char} k = 0$, this object is in $\mathrm{DM}_{\mathrm{gm}}^{\mathrm{eff}}$ by [86, §4.1].

Convention. In this subsection, “scheme” means separated k -scheme of finite type if $\text{char } k = 0$ and smooth (separated) k -scheme of finite type if $\text{char } k > 0$.

8.1.1. **Definition.** We define the *homological Albanese complex* of X by

$$\text{LAlb}(X) := \text{LAlb}(M(X)).$$

Define, for $i \in \mathbb{Z}$

$$\text{L}_i\text{Alb}(X) := {}_tH_i(\text{LAlb}(X))$$

the 1-motives with cotorsion (see Definition 1.8.1 and Notation 1.8.7) determined by the homology of the Albanese complex.

The functor LAlb has the following properties, easily deduced from [86, 2.2]:

8.1.a. *Homotopy invariance.* For any scheme X the map

$$\text{LAlb}(X \times \mathbb{A}^1) \rightarrow \text{LAlb}(X)$$

is an isomorphism, thus

$$\text{L}_i\text{Alb}(X \times \mathbb{A}^1) \xrightarrow{\sim} \text{L}_i\text{Alb}(X)$$

for all $i \in \mathbb{Z}$.

8.1.b. *Mayer-Vietoris.* For a scheme X and an open covering $X = U \cup V$ there is a distinguished triangle

$$\begin{array}{ccc} \text{LAlb}(U \cap V) & \longrightarrow & \text{LAlb}(U) \oplus \text{LAlb}(V) \\ +1 \nearrow & & \swarrow \\ & \text{LAlb}(X) & \end{array}$$

and therefore a long exact sequence of 1-motives

$$\cdots \rightarrow \text{L}_i\text{Alb}(U \cap V) \rightarrow \text{L}_i\text{Alb}(U) \oplus \text{L}_i\text{Alb}(V) \rightarrow \text{L}_i\text{Alb}(X) \rightarrow \cdots$$

8.1.c. *Tate twists.* If X is a smooth scheme and $n > 0$, then

$$\text{Tot LAlb}(M(X)(n)) = \begin{cases} 0 & \text{if } n > 1 \\ M(\pi_0(X))(1) & \text{if } n = 1 \end{cases}$$

where $\pi_0(X)$ is the scheme of constants of X , see Definition 2.5.1. Indeed

$$\begin{aligned} \text{Tot LAlb}(M(X)(n)) &= \underline{\text{Hom}}_{\text{ét}} \alpha^*(\underline{\text{Hom}}_{\text{Nis}}(M(X)(n), \mathbb{Z}(1)), \mathbb{Z}(1)) \\ &= \underline{\text{Hom}}_{\text{ét}} \alpha^*(\underline{\text{Hom}}_{\text{Nis}}(M(X)(n-1), \mathbb{Z}), \mathbb{Z}(1)) \end{aligned}$$

by the cancellation theorem [87]. Now

$$\underline{\mathrm{Hom}}_{\mathrm{Nis}}(M(X)(n-1), \mathbb{Z}) = \begin{cases} 0 & \text{if } n > 1 \\ \underline{\mathrm{Hom}}_{\mathrm{Nis}}(M(\pi_0(X)), \mathbb{Z}) & \text{if } n = 1. \end{cases}$$

The last formula should follow from [48, Lemma 2.1 a)] but the formulation there is wrong; however, the formula immediately follows from the argument in the proof of *loc. cit.*, *i.e.* considering the Zariski cohomology of X with coefficients in the flasque sheaf \mathbb{Z} .

This gives

$$(8.1) \quad \mathrm{L}_i \mathrm{Alb}(M(X)(1)) = \begin{cases} [0 \rightarrow R_{\pi_0(X)/k} \mathbb{G}_m] & \text{if } i = 0 \\ 0 & \text{else} \end{cases}$$

where $R_{L/k}(-)$ is Weil's restriction of scalars.

From this computation we formally deduce:

8.1.2. Proposition. *For any $M \in \mathrm{DM}_{\mathrm{gm}}^{\mathrm{eff}}$, we have*

- $\mathrm{LAlb}(M(n)) = 0$ for $n \geq 2$;
- $\mathrm{LAlb}(M(1))$ is a complex of toric 1-motives. □

8.1.d. *Gysin.* Let Z be a closed smooth subscheme purely of codimension n of a smooth scheme X , and let $U = X - Z$. Then

$$\mathrm{LAlb}(U) \xrightarrow{\sim} \mathrm{LAlb}(X) \text{ if } n > 1$$

and if $n = 1$ we have an exact triangle

$$\mathrm{LAlb}(U) \rightarrow \mathrm{LAlb}(X) \rightarrow [0 \rightarrow R_{\pi_0(Z)/k} \mathbb{G}_m][2] \rightarrow \mathrm{LAlb}(U)[1]$$

hence a long exact sequence of 1-motives

$$\begin{aligned} 0 \rightarrow \mathrm{L}_2 \mathrm{Alb}(U) \rightarrow \mathrm{L}_2 \mathrm{Alb}(X) \rightarrow [0 \rightarrow R_{\pi_0(Z)/k} \mathbb{G}_m] \\ \rightarrow \mathrm{L}_1 \mathrm{Alb}(U) \rightarrow \mathrm{L}_1 \mathrm{Alb}(X) \rightarrow 0 \end{aligned}$$

and an isomorphism $\mathrm{L}_0 \mathrm{Alb}(U) \rightarrow \mathrm{L}_0 \mathrm{Alb}(X)$.

8.1.e. *Blow ups.* If X is a scheme and $Z \subseteq X$ is a closed subscheme, denote by $p : \tilde{X} \rightarrow X$ a proper surjective morphism such that $p^{-1}(X - Z) \rightarrow X - Z$ is an isomorphism, *e.g.* the blow up of X at Z . Then there is a distinguished triangle

$$\begin{array}{ccc} \mathrm{LAlb}(\tilde{Z}) & \longrightarrow & \mathrm{LAlb}(\tilde{X}) \oplus \mathrm{LAlb}(Z) \\ +1 \swarrow & & \searrow \\ & \mathrm{LAlb}(X) & \end{array}$$

with $\tilde{Z} = p^{-1}(Z)$, yielding a long exact sequence of 1-motives

$$(8.2) \quad \cdots \rightarrow L_i \mathrm{Alb}(\tilde{Z}) \rightarrow L_i \mathrm{Alb}(\tilde{X}) \oplus L_i \mathrm{Alb}(Z) \rightarrow L_i \mathrm{Alb}(X) \rightarrow \cdots$$

If X and Z are smooth, we get (using [86, Prop. 3.5.3] and the above)

$$L\mathrm{Alb}(\tilde{X}) = L\mathrm{Alb}(X) \oplus [0 \rightarrow R_{\pi_0(Z)/k} \mathbb{G}_m][2]$$

and corresponding formulas for homology.

8.1.f. *Albanese map.* If X is a scheme we have the natural map (5.2) in $\mathrm{DM}_{\mathrm{gm}, \mathrm{ét}}^{\mathrm{eff}}$

$$(8.3) \quad a_X : \alpha^* M(X) \rightarrow \mathrm{Tot} L\mathrm{Alb}(X)$$

inducing homomorphisms on étale motivic cohomology

$$\mathrm{Hom}(M(X), \mathbb{Z}(1)[j]) \rightarrow \mathrm{Hom}(L\mathrm{Alb}(X), \mathbb{G}_m[j-1])$$

which are isomorphisms rationally by Theorem 6.2.1.

8.2. **Cohomological Pic.** Dual to 8.1.1 we set:

8.2.1. **Definition.** Define the *cohomological Picard complex* of X by

$$\mathrm{RPic}(X) := \mathrm{RPic}(M(X)).$$

Define, for $i \in \mathbb{Z}$

$$\mathrm{R}^i \mathrm{Pic}(X) := {}^t H^i(\mathrm{RPic}(X))$$

the 1-motives with torsion determined by the cohomology of the Picard complex (see Notation 1.8.7).

The functor RPic has similar properties to $L\mathrm{Alb}$, deduced by duality. Homotopical invariance, Mayer-Vietoris, Gysin and the distinguished triangle for abstract blow-ups are clear, and moreover we have

$$\mathrm{RPic}(M(X)(n)) = \begin{cases} 0 & \text{if } n > 1 \\ [\mathbb{Z}\pi_0(X) \rightarrow 0] & \text{if } n = 1. \end{cases}$$

We also have that $\mathrm{RPic}(X) = L\mathrm{Alb}(X)^\vee$, hence

$$\mathrm{R}^i \mathrm{Pic}(X) = L_i \mathrm{Alb}(X)^\vee.$$

We shall complete §8.1.d by

8.2.a. *RPic and LAlb with supports.* Let $X \in Sm(k)$, U a dense open subset of X and $Z = X - U$ (reduced structure). In DM, we have the *motive with supports* $M^Z(X)$ fitting in an exact triangle

$$M(U) \rightarrow M(X) \rightarrow M^Z(X) \xrightarrow{+1}$$

hence the cohomological complex with supports

$$\mathrm{RPic}_Z(X) := \mathrm{RPic}(M^Z(X))$$

fitting in an exact triangle

$$\mathrm{RPic}_Z(X) \rightarrow \mathrm{RPic}(X) \rightarrow \mathrm{RPic}(U) \xrightarrow{+1}.$$

Dually, we have $\mathrm{LAlb}^Z(X) := \mathrm{LAlb}(M^Z(X))$.

8.2.2. **Lemma.** *If $\dim X = d$, $\mathrm{RPic}_Z(X) \simeq [\underline{CH}_{d-1}(Z) \rightarrow 0][-2]$, where $\underline{CH}_{d-1}(Z)$ is the lattice corresponding to the Galois module $CH_{d-1}(Z_{\bar{k}})$.*

Note that $CH_{d-1}(Z_{\bar{k}})$ is simply the free abelian group with basis the irreducible components of $Z_{\bar{k}}$ which are of codimension 1 in $X_{\bar{k}}$.

Proof. This follows readily from the exact sequence

$$\begin{aligned} 0 \rightarrow \Gamma(X_{\bar{k}}, \mathbb{G}_m) &\rightarrow \Gamma(U_{\bar{k}}, \mathbb{G}_m) \rightarrow CH_{d-1}(Z_{\bar{k}}) \\ &H^1(X_{\bar{k}}, \mathbb{G}_m) \rightarrow H^1(U_{\bar{k}}, \mathbb{G}_m) \rightarrow 0. \end{aligned}$$

□

8.3. **Relative LAlb and RPic.** For $f : Y \rightarrow X$ a map of schemes we let $M(X, Y)$ denote the cone of $C_*(Y) \rightarrow C_*(X)$. Note that for a closed embedding f in a proper scheme X we have an isomorphism $M(X, Y) \xrightarrow{\sim} M^c(X - Y)$.

We denote by $\mathrm{LAlb}(X, Y)$ and $\mathrm{RPic}(X, Y)$ the resulting complexes of 1-motives.

9. BOREL-MOORE VARIANTS

9.1. **The Borel-Moore Albanese complex.** Let $p : X \rightarrow \mathrm{Spec} k$ be a scheme of finite type over a field k which admits resolution of singularities. Recall that the motive with compact support of X , denoted $M^c(X) \in \mathrm{DM}_-^{\mathrm{eff}}$, has also been defined in [86, Sect. 4]. It is the Suslin complex C_* of the representable presheaf with transfers $L^c(X)$ given by quasi-finite correspondences. Since finite implies quasi-finite we have a canonical map $M(X) \rightarrow M^c(X)$ which is an isomorphism if X is proper over k .

In general, $M^c : Sch^c(k) \rightarrow \mathrm{DM}_{\mathrm{gm}}^{\mathrm{eff}}$ is a covariant functor from the category of schemes of finite type over k and proper maps between them.

9.1.1. Definition. We define the *Borel-Moore Albanese complex* of X by

$$\mathrm{LAlb}^c(X) := \mathrm{LAlb}(M^c(X)).$$

Define, for $i \in \mathbb{Z}$

$$\mathrm{L}_i \mathrm{Alb}^c(X) := {}_t H_i(\mathrm{LAlb}^c(X))$$

the 1-motivic homology of this complex.

Note that we have the following properties:

9.1.a. Functoriality. The functor $X \mapsto \mathrm{LAlb}^c(X)$ is covariant for proper maps and contravariant with respect to flat morphisms of relative dimension zero, for example étale morphisms. We have a canonical, covariantly functorial map

$$\mathrm{LAlb}(X) \rightarrow \mathrm{LAlb}^c(X)$$

which is an isomorphism if X is proper.

9.1.b. Localisation triangle. For any closed subscheme Y of a scheme X we have a triangle

$$\begin{array}{ccc} \mathrm{LAlb}^c(Y) & \longrightarrow & \mathrm{LAlb}^c(X) \\ +1 \swarrow & & \searrow \\ & \mathrm{LAlb}^c(X - Y) & \end{array}$$

and therefore a long exact sequence of 1-motives

$$(9.1) \quad \cdots \rightarrow \mathrm{L}_i \mathrm{Alb}^c(Y) \rightarrow \mathrm{L}_i \mathrm{Alb}^c(X) \rightarrow \mathrm{L}_i \mathrm{Alb}^c(X - Y) \rightarrow \mathrm{L}_{i-1} \mathrm{Alb}^c(Y) \rightarrow \cdots$$

In particular, let X be a scheme obtained by removing a divisor Y from a proper scheme \overline{X} , i.e. $X = \overline{X} - Y$. Then

$$\begin{aligned} \cdots \rightarrow \mathrm{L}_1 \mathrm{Alb}(Y) \rightarrow \mathrm{L}_1 \mathrm{Alb}(\overline{X}) \rightarrow \mathrm{L}_1 \mathrm{Alb}^c(X) \rightarrow \mathrm{L}_0 \mathrm{Alb}(Y) \\ \rightarrow \mathrm{L}_0 \mathrm{Alb}(\overline{X}) \rightarrow \mathrm{L}_0 \mathrm{Alb}^c(X) \rightarrow \cdots \end{aligned}$$

9.1.c. Albanese map. We have the following natural map (5.2)

$$a_X^c : \alpha^* M^c(X) \rightarrow \mathrm{Tot} \mathrm{LAlb}^c(X)$$

which is an isomorphism if $\dim(X) \leq 1$. In general, for any X , a_X^c induces an isomorphism on motivic cohomology with compact supports, i.e. $H_c^j(X, \mathbb{Q}(1)) = \mathrm{Hom}(\mathrm{LAlb}^c(X), \mathbb{G}_m[j-1]) \otimes \mathbb{Q}$.

9.2. Cohomological Albanese complex.

9.2.1. Lemma. *Suppose $p = 1$ (i.e. $\text{char } k = 0$), and let $n \geq 0$. For any X of dimension $\leq n$, the motive $M(X)^*(n)[2n]$ is effective. (Here, contrary to the rest of the paper, $M(X)^*$ denotes the “usual” dual $\text{Hom}(M(X), \mathbb{Z})$ in DM_{gm} .)*

Proof. First assume X irreducible. Let $\tilde{X} \rightarrow X$ be a resolution of singularities of X . With notation as in §8.1.e, we have an exact triangle

$$M(X)^*(n) \rightarrow M(\tilde{X})^*(n) \oplus M(Z)^*(n) \oplus M(\tilde{Z})^*(n) \xrightarrow{+1}.$$

Since \tilde{X} is smooth, $M(\tilde{X})^*(n) \simeq M^c(X)[-2n]$ is effective by [86, Th. 4.3.2]; by induction on n , so are $M(Z)^*(n)$ and $M(\tilde{Z})^*(n)$ and therefore $M(X)^*(n)$ is effective.

In general, let X_1, \dots, X_r be the irreducible components of X . Suppose $r \geq 2$ and let $Y = X_2 \cup \dots \cup X_r$: since (X_1, Y) is a cdh cover of X , we have an exact triangle

$$M(X)^*(n) \rightarrow M(X_1)^*(n) \oplus M(Y)^*(n) \oplus M(X_1 \cap Y)^*(n) \xrightarrow{+1}.$$

The same argument then shows that $M(X)^*(n)$ is effective, by induction on r . \square

We can therefore apply our functor LAlb and obtain another complex $\text{LAlb}(M(X)^*(n)[2n])$ of 1-motives. If X is smooth this is just the Borel-Moore Albanese.

9.2.2. Definition. We define the *cohomological Albanese complex* of a scheme X of dimension n by

$$\text{LAlb}^*(X) := \text{LAlb}(M(X^{(n)})^*(n)[2n])$$

where $X^{(n)}$ is the union of the n -dimensional components of X . Define, for $i \in \mathbb{Z}$

$$\text{L}_i \text{Alb}^*(X) := {}_t H_i(\text{LAlb}^*(X))$$

the 1-motivic homology of this complex.

9.2.3. Lemma. *a) If Z_1, \dots, Z_n are the irreducible components of dimension n of X , then the cone of the natural map*

$$\text{LAlb}^*(X) \rightarrow \bigoplus \text{LAlb}^*(Z_i)$$

is a complex of groups of multiplicative type.

b) If X is integral and \tilde{X} is a desingularisation of X , then the cone of the natural map

$$\text{LAlb}^*(X) \rightarrow \text{LAlb}^*(\tilde{X})$$

is a complex of groups of multiplicative type.

Proof. a) and b) follow from dualising the abstract blow-up exact triangles of [86, 2.2] and applying Proposition 8.1.2. \square

9.3. Compactly supported and homological Pic. We now consider the dual complexes of the Borel-Moore and cohomological Albanese.

9.3.1. Definition. Define the *compactly supported Picard complex* of any scheme X by

$$\mathrm{RPic}^c(X) := \mathrm{RPic}(M^c(X))$$

and the *homological Picard complex* of an equidimensional scheme X of dimension n by

$$\mathrm{RPic}^*(X) := \mathrm{RPic}(M(X)^*[n][2n]).$$

Denote $\mathrm{R}^i\mathrm{Pic}^c(X) := {}^tH^i(\mathrm{RPic}^c(X))$ and $\mathrm{R}^i\mathrm{Pic}^*(X) := {}^tH^i(\mathrm{RPic}^*(X))$ the 1-motives with torsion determined by the homology of these Picard complexes.

Recall that $\mathrm{RPic}^c(X) = \mathrm{RPic}(X)$ if X is proper and $\mathrm{RPic}^c(X) = \mathrm{RPic}^*(X)$ if X is smooth.

9.4. Topological invariance. To conclude this section and the previous one, we note the following useful

9.4.1. Lemma. *Suppose that $f : Y \rightarrow X$ is a universal topological homeomorphism, in the sense that $1_U \times f : U \times Y \rightarrow U \times X$ is a homeomorphism of topological spaces for any smooth U (in particular f is proper). Then f induces isomorphisms $\mathrm{LAlb}(Y) \xrightarrow{\sim} \mathrm{LAlb}(X)$, $\mathrm{RPic}(X) \xrightarrow{\sim} \mathrm{RPic}(Y)$, $\mathrm{LAlb}^c(Y) \xrightarrow{\sim} \mathrm{LAlb}^c(X)$ and $\mathrm{RPic}^c(X) \xrightarrow{\sim} \mathrm{RPic}^c(Y)$. Similarly, $\mathrm{LAlb}^*(X) \xrightarrow{\sim} \mathrm{LAlb}^*(Y)$ and $\mathrm{RPic}^*(Y) \xrightarrow{\sim} \mathrm{RPic}^*(X)$. This applies in particular to $Y =$ the semi-normalisation of X .*

Proof. It suffices to notice that f induces isomorphisms $L(Y) \xrightarrow{\sim} L(X)$ and $L^c(Y) \xrightarrow{\sim} L^c(X)$, since by definition these sheaves only depend on the underlying topological structures. \square

Lemma 9.4.1 implies that in order to compute $\mathrm{LAlb}(X)$, etc., we may always assume X semi-normal if we wish so.

10. COMPUTING $\mathrm{LAlb}(X)$ AND $\mathrm{RPic}(X)$ FOR SMOOTH X

10.1. The Albanese scheme. Let X be a reduced k -scheme of finite type. “Recall” ([71, Sect. 1], [81]) the Albanese scheme $\mathcal{A}_{X/k}$ fitting in the following extension

$$(10.1) \quad 0 \rightarrow \mathcal{A}_{X/k}^0 \rightarrow \mathcal{A}_{X/k} \rightarrow \mathbb{Z}[\pi_0(X)] \rightarrow 0$$

where $\mathcal{A}_{X/k}^0$ is Serre’s generalised Albanese semi-abelian variety, and $\pi_0(X)$ is the scheme of constants of X viewed as an étale sheaf on $Sm(k)$.¹² In particular,

$$\mathcal{A}_{X/k} \in \mathrm{AbS} \text{ (see Definition 3.7.1).}$$

There is a canonical morphism

$$(10.2) \quad \bar{a}_X : X \rightarrow \mathcal{A}_{X/k}$$

which is universal for morphisms from X to group schemes of the same type.

For the existence of $\mathcal{A}_{X/k}$, the reference [71, Sect. 1] is sufficient if X is a variety (integral separated k -scheme of finite type), hence if X is normal (for example smooth): this will be sufficient in this section. For the general case, see §11.1.

We shall denote the object $\mathrm{Tot}^{-1}(\underline{\mathcal{A}}_{X/k}) \in D^b(\mathcal{M}_1[1/p])$ simply by $\mathcal{A}_{X/k}$. As seen in Lemma 3.10.1, we have

$$H_i(\mathcal{A}_{X/k}) = \begin{cases} [\mathbb{Z}[\pi_0(X)] \rightarrow 0] & \text{for } i = 0 \\ [0 \rightarrow \mathcal{A}_{X/k}^0] & \text{for } i = 1 \\ 0 & \text{for } i \neq 0, 1. \end{cases}$$

10.2. The main theorem. Suppose X smooth. Via (1.1), (10.2) induces a composite map

$$(10.3) \quad M_{\mathrm{\acute{e}t}}(X) \rightarrow M_{\mathrm{\acute{e}t}}(\mathcal{A}_{X/k}) \rightarrow \mathcal{A}_{X/k}.$$

Theorem 4.5.3 gives:

10.2.1. Lemma. *We have an exact triangle*

$$\mathbb{Z}[\pi_0(X)]^*[0] \rightarrow \underline{\mathrm{Hom}}(\mathcal{A}_{X/k}, \mathbb{Z}(1)) \rightarrow (\mathcal{A}_{X/k}^0)^*[-2] \xrightarrow{+1} \square$$

By Lemma 10.2.1, the map

$$\underline{\mathrm{Hom}}(\mathcal{A}_{X/k}, \mathbb{Z}(1)) \rightarrow \underline{\mathrm{Hom}}(M_{\mathrm{\acute{e}t}}(X), \mathbb{Z}(1))$$

deduced from (10.3) factors into a map

$$(10.4) \quad \underline{\mathrm{Hom}}(\mathcal{A}_{X/k}, \mathbb{Z}(1)) \rightarrow \tau_{\leq 2} \underline{\mathrm{Hom}}(M_{\mathrm{\acute{e}t}}(X), \mathbb{Z}(1)).$$

¹²In the said references, $\mathcal{A}_{X/k}^0$ is denoted by Alb_X and $\mathcal{A}_{X/k}$ is denoted by $\widetilde{\mathrm{Alb}}_X$.

Applying Proposition 4.5.1 and Lemma 5.1.2, we therefore get a canonical map in $D^b(\mathcal{M}_1[1/p])$

$$(10.5) \quad \mathrm{LAlb}(X) \rightarrow \mathcal{A}_{X/k}.$$

10.2.2. Theorem. *Suppose X smooth. Then the map (10.5) sits in an exact triangle*

$$[0 \rightarrow \mathrm{NS}_{X/k}^*[2] \rightarrow \mathrm{LAlb}(X) \rightarrow \mathcal{A}_{X/k} \xrightarrow{+1}$$

where $\mathrm{NS}_{X/k}^*$ denotes the group of multiplicative type dual to $\mathrm{NS}_{X/k}$ (cf. Definition 3.5.1 and Proposition 3.5.2).

This theorem says in particular that, on the object $\mathrm{LAlb}(X)$, the motivic t -structure and the homotopy t -structure are compatible in a strong sense.

10.2.3. Corollary. *For X smooth over k we have*

$$\mathrm{L}_i \mathrm{Alb}(X) = \begin{cases} [\mathbb{Z}[\pi_0(X)] \rightarrow 0] & \text{if } i = 0 \\ [0 \rightarrow \mathcal{A}_{X/k}^0] & \text{if } i = 1 \\ [0 \rightarrow \mathrm{NS}_{X/k}^*] & \text{if } i = 2 \\ 0 & \text{otherwise.} \end{cases}$$

10.2.4. Corollary. *For X smooth, $\mathrm{L}_1 \mathrm{Alb}(X)$ is isomorphic to the homological Albanese 1-motive $\mathrm{Alb}^-(X)$ of [8].* \square

10.3. Reformulation of Theorem 10.2.2. It is sufficient to get an exact triangle after application of $D_{\leq 1} \circ \mathrm{Tot}$, so that we have to compute the cone of the morphism (10.4) in $\mathrm{DM}_{\mathrm{gm}, \mathrm{ét}}^{\mathrm{eff}}$. We shall use:

10.3.1. Lemma. *For $\mathcal{F} \in \mathrm{HI}_{\mathrm{ét}}^s$, the morphism b of Proposition 3.1.1 is induced by (1.1).*

Proof. This is clear by construction, since $\mathrm{Hom}_{\mathrm{DM}_{\mathrm{ét}}}(M(G), \mathcal{F}[1]) = H_{\mathrm{ét}}^1(G, \mathcal{F})$ [86, Prop. 3.3.1]. \square

Taking the cohomology sheaves of (10.4), we get morphisms

$$(10.6) \quad \mathcal{H}om(\mathcal{A}_{X/k}, \mathbb{G}_m) \rightarrow p_* \mathbb{G}_{m,X}$$

$$(10.7) \quad f : \mathcal{E}xt(\mathcal{A}_{X/k}, \mathbb{G}_m) \xrightarrow{b} \underline{\mathrm{Pic}}_{\mathcal{A}_{X/k}/k} \xrightarrow{\bar{a}^*} \underline{\mathrm{Pic}}_{X/k}$$

where in (10.7), b corresponds to the map of Proposition 3.1.1 thanks to Lemma 10.3.1. Thanks to Proposition 3.6.1, Theorem 10.2.2 is then equivalent to the following

10.3.2. Theorem. *Suppose k algebraically closed; Then*

- a) (10.6) yields an isomorphism $\mathrm{Hom}(\mathcal{A}_{X/k}, \mathbb{G}_m) \xrightarrow{\sim} \Gamma(X, \mathbb{G}_m)$.*
- b) (10.7) defines a short exact sequence*

$$(10.8) \quad 0 \rightarrow \mathrm{Ext}(\underline{\mathcal{A}}_{X/k}, \mathbb{G}_m) \xrightarrow{f} \mathrm{Pic}(X) \xrightarrow{e} \mathrm{NS}(X) \rightarrow 0$$

where e is the natural map.

Before proving Theorem 10.3.2, it is convenient to prove Lemma 10.3.3 below. Let $A_{X/k}$ be the abelian part of $\mathcal{A}_{X/k}^0$; then the sheaf $\mathcal{E}xt(\underline{\mathcal{A}}_{X/k}, \mathbb{G}_m)$ is represented by the dual abelian variety $A_{X/k}^*$. Composing with the map f of (10.7), we get a map of 1-motivic sheaves

$$(10.9) \quad \underline{A}_{X/k}^* \rightarrow \underline{\mathrm{Pic}}_{X/k}.$$

10.3.3. Lemma. *The map (10.9) induces an isogeny in SAb*

$$A_{X/k}^* \twoheadrightarrow \gamma(\underline{\mathrm{Pic}}_{X/k}) = \underline{\mathrm{Pic}}_{X/k}^0$$

where γ is the adjoint functor appearing in Theorem 3.2.4 a).

Proof. We proceed in 3 steps:

- (1) The lemma is true if X is smooth projective: this follows from the representability of Pic_X^0 and the duality between the Picard and the Albanese varieties.
- (2) Let $j : U \rightarrow X$ be an open immersion: then the lemma is true for X if and only if it is true for U . This is clear since $\underline{\mathrm{Pic}}_{X/k} \rightarrow \underline{\mathrm{Pic}}_{U/k}$ is an epimorphism with discrete kernel.
- (3) Let $\varphi : Y \rightarrow X$ be an étale covering. If the lemma is true for Y , then it is true for X . This follows from the existence of transfer maps $\varphi_* : A_{Y/k}^* \rightarrow A_{X/k}^*$, $\underline{\mathrm{Pic}}_{Y/k} \rightarrow \underline{\mathrm{Pic}}_{X/k}$ commuting with the map of the lemma, plus the usual transfer argument.

We conclude by de Jong's theorem [47, Th. 4.1]. \square

10.4. Proof of Theorem 10.3.2. We may obviously suppose that X is irreducible.

a) is obvious from the universal property of $\mathcal{A}_{X/k}$. For b) we proceed in two steps:

- (1) Verification of $ef = 0$.
- (2) Proof that the sequence is exact.

(1) As above, let $A = A_{X/k}$ be the abelian part of $\mathcal{A}_{X/k}^0$. In the diagram

$$\mathrm{Ext}(A, \mathbb{G}_m) \rightarrow \mathrm{Ext}(\mathcal{A}_{X/k}^0, \mathbb{G}_m) \leftarrow \mathrm{Ext}(\mathcal{A}_{X/k}, \mathbb{G}_m)$$

the first map is surjective and the second map is an isomorphism, hence we get a surjective map

$$v : \text{Ext}(A, \mathbb{G}_m) \rightarrow \text{Ext}(\mathcal{A}_{X/k}, \mathbb{G}_m).$$

Choose a rational point $x \in X(\bar{k})$. We have a diagram

$$\begin{array}{ccc} \text{Ext}(A, \mathbb{G}_m) & \xrightarrow{v} & \text{Ext}(\mathcal{A}_{X/k}, \mathbb{G}_m) \\ a \downarrow & & f \downarrow \\ \text{Pic}(A) & \xrightarrow{x^*} & \text{Pic}(X) \\ c \downarrow & & e \downarrow \\ \text{NS}(A) & \xrightarrow{x^*} & \text{NS}(X) \end{array}$$

in which

- (i) a is given by [77, p. 170, prop. 5 and 6] (or by Proposition 3.1.1).
- (ii) $ca = 0$ (ibid., p. 184, th. 6;).
- (iii) x^* is induced by the “canonical” map $X \rightarrow A$ sending x to 0.

Lemma 3.1.6 applied to $G = \mathcal{A}_{X/k}$ implies that the top square commutes (the bottom one trivially commutes too). Moreover, since v is surjective and $ca = 0$, we get $ef = 0$.

(2) In the sequence (10.8), the surjectivity of e is clear. Let us prove the injectivity of f : suppose that $f(\mathcal{E})$ is trivial. In the pull-back diagram

$$\begin{array}{ccc} \bar{a}^* \mathcal{E} & \xrightarrow{\pi'} & X \\ \bar{a}' \downarrow & & \bar{a} \downarrow \\ \mathcal{E} & \xrightarrow{\pi} & \mathcal{A}_{X/k} \end{array}$$

π' has a section σ' . Observe that \mathcal{E} is a locally semi-abelian scheme: by the universal property of $\mathcal{A}_{X/k}$, the morphism $\bar{a}'\sigma'$ factors canonically through \bar{a} . In other words, there exists $\sigma : \mathcal{A}_{X/k} \rightarrow \mathcal{E}$ such that $\bar{a}'\sigma' = \sigma\bar{a}$. Then

$$\pi\sigma\bar{a} = \pi\bar{a}'\sigma' = \bar{a}\pi'\sigma' = \bar{a}$$

hence $\pi\sigma = 1$ by reapplying the universal property of $\mathcal{A}_{X/k}$, and \mathcal{E} is trivial. Finally, exactness in the middle follows immediately from Proposition 3.5.2 and Lemma 10.3.3. \square

10.4.1. Corollary. *The isogeny of Lemma 10.3.3 is an isomorphism.*

Proof. This follows from the injectivity of f in (10.8). \square

10.5. An application.

10.5.1. Corollary. *Let X be a smooth k -variety of dimension d , U a dense open subset and $Z = X - U$ (reduced structure). Then the morphism $\mathcal{A}_{U/k} \rightarrow \mathcal{A}_{X/k}$ is epi; its kernel $T_{X/U,k}$ is a torus whose character group fits into a short exact sequence*

$$0 \rightarrow T_{X/U,k}^* \rightarrow \underline{\mathrm{CH}}_{d-1}(Z) \rightarrow \mathrm{NS}_Z(X) \rightarrow 0$$

where $\underline{\mathrm{CH}}_{d-1}(Z)$ is as in Lemma 8.2.2 and $\mathrm{NS}_Z(X) = \mathrm{Ker}(\mathrm{NS}(X) \rightarrow \mathrm{NS}(U))$.

Proof. To see that $\mathcal{A}_{U/k} \rightarrow \mathcal{A}_{X/k}$ is epi with kernel of multiplicative type, it is sufficient to see that $\pi_0(\mathcal{A}_{U/k}) \xrightarrow{\sim} \pi_0(\mathcal{A}_{X/k})$ and that $\mathcal{A}_{U/k} \xrightarrow{\sim} \mathcal{A}_{X/k}$. The first isomorphism is obvious and the second one follows from [63, Th. 3.1]. The characterisation of $T_{X/U,k}$ is then an immediate consequence of Theorem 10.2.2 and Lemma 8.2.2; in particular, it is a torus. \square

10.6. $\mathrm{RPic}(X)$. Recall that for X smooth projective $\mathcal{A}_{X/k}^0 = \mathrm{Alb}(X)$ is the classical Albanese abelian variety $\mathrm{Alb}(X)$. In the case where X is obtained by removing a divisor Y from a smooth proper scheme \overline{X} , $\mathcal{A}_{X/k}^0$ can be described as follows (cf. [8]). Consider the (cohomological Picard) 1-motive $\mathrm{Pic}^+(X) := [\mathrm{Div}_Y^0(\overline{X}) \rightarrow \mathrm{Pic}^0(\overline{X})]$: its Cartier dual is $\mathcal{A}_{X/k}^0$ which can be represented as a torus bundle

$$0 \rightarrow T_{\overline{X}/X,k} \rightarrow \mathcal{A}_{X/k}^0 \rightarrow \mathrm{Alb}(\overline{X}) \rightarrow 0$$

where $T_{\overline{X}/X,k}$ has character group $\mathrm{Div}_Y^0(\overline{X})$ according to Corollary 10.5.1.

From the previous remarks and Corollary 10.2.4, we deduce:

10.6.1. Corollary. *If X is smooth, $\mathrm{R}^1\mathrm{Pic}(X)$ is isomorphic to the 1-motive $\mathrm{Pic}^+(X)$ of [8] (the Cartier dual of $\mathrm{Alb}^-(X)$). If \overline{X} is a smooth compactification of X , then*

$$\mathrm{R}^i\mathrm{Pic}(X) = \begin{cases} [0 \rightarrow \mathbb{Z}[\pi_0(X)]^*] & \text{if } i = 0 \\ [\mathrm{Div}_Y^0(\overline{X}) \rightarrow \mathrm{Pic}^0(\overline{X})] & \text{if } i = 1 \\ [\mathrm{NS}(X) \rightarrow 0] & \text{if } i = 2 \\ 0 & \text{otherwise} \end{cases}$$

where $Y = \overline{X} - X$.

11. 1-MOTIVIC HOMOLOGY AND COHOMOLOGY OF SINGULAR SCHEMES

11.1. $\mathcal{A}_{X/k}$ **for** $X \in \text{Sch}(k)$. In this subsection, we extend the construction of $\mathcal{A}_{X/k}$ to arbitrary reduced k -schemes of finite type, starting from the case where X is integral (which is treated in [71, Sect. 1]). So far, k may be of any characteristic.

To make the definition clear:

11.1.1. **Definition.** Let $X \in \text{Sch}(k)$. We say that $\mathcal{A}_{X/k}$ *exists* if the functor

$$\begin{aligned} \text{AbS} &\rightarrow \text{Ab} \\ G &\mapsto G(X) \end{aligned}$$

is corepresentable.

First note that $\mathcal{A}_{X/k}$ does not exist (as a semi-abelian scheme, at least) if X is not reduced. For example, for $X = \text{Spec } k[\varepsilon]$ with $\varepsilon^2 = 0$, we have an exact sequence

$$0 \rightarrow \mathbb{G}_a(k) \rightarrow \text{Map}_k(X, \mathbb{G}_m) \rightarrow \mathbb{G}_m(k) \rightarrow 0$$

which cannot be described by $\text{Hom}(\mathcal{A}, \mathbb{G}_m)$ for any semi-abelian scheme \mathcal{A} .

On the other hand, $M(X) = M(X_{\text{red}})$ for any $X \in \text{Sch}(k)$, where X_{red} is the reduced subscheme of X (see proof of Lemma 9.4.1), so we are naturally led to neglect nonreduced schemes.

11.1.2. **Lemma.** *Let $Z \in \text{Sch}(k)$, $G \in \text{AbS}$ and $f_1, f_2 : Z \rightrightarrows G$ two morphisms which coincide on the underlying topological spaces (thus, $f_1 = f_2$ if Z is reduced). Then there exists a largest quotient \overline{G} of G such that $\pi_0(G) \xrightarrow{\sim} \pi_0(\overline{G})$ and the two compositions*

$$Z \rightrightarrows G \rightarrow \overline{G}$$

coincide.

Proof. The set S of such quotients \overline{G} is in one-to-one correspondence with the set of closed subgroups $H^0 \subseteq G^0$. Clearly $\pi_0(G) \in S$, and if $\overline{G}_1 = G/H_1^0 \in S$, $\overline{G}_2 = G/H_2^0 \in S$, then $G_3 = G/(H_1^0 \cap H_2^0) \in S$. Therefore S has a smallest element, since it is Artinian (compare proof of Proposition 3.6.5). \square

11.1.3. **Proposition.** $\mathcal{A}_{X/k}$ *exists for any reduced $X \in \text{Sch}(k)$.*

Proof. When X is integral, this is [71, Sect. 1]. Starting from this case, we argue by induction on $\dim X$. Let Z_1, \dots, Z_n be the irreducible components of X and $Z_{ij} = Z_i \cap Z_j$.

By induction, $\mathcal{A}_{ij} := \mathcal{A}_{(Z_{ij})_{\text{red}}/k}$ exists for any (i, j) . Consider

$$\mathcal{A} = \text{Coker} \left(\bigoplus \mathcal{A}_{ij} \rightarrow \bigoplus \mathcal{A}_i \right)$$

with $\mathcal{A}_i = \mathcal{A}_{Z_i/k}$. Let $Z = \coprod Z_{ij}$ and $f_1, f_2 : Z \rightrightarrows \coprod Z_i$ be the two inclusions: the compositions $f_1, f_2 : Z \rightrightarrows \mathcal{A}$ verify the hypothesis of Lemma 11.1.2. Hence there is a largest quotient \mathcal{A}' of \mathcal{A} with $\pi_0(\mathcal{A}) \xrightarrow{\sim} \pi_0(\mathcal{A}')$, equalising f_1 and f_2 . Then the composition

$$\coprod Z_i \rightarrow \bigoplus \mathcal{A}_i \rightarrow \mathcal{A}'$$

glues down to a morphism $X \rightarrow \mathcal{A}'$. It is clear that $\mathcal{A}' = \mathcal{A}_{X/k}$ since, for any commutative group scheme G , the sequence

$$0 \rightarrow \text{Map}_k(X, G) \rightarrow \bigoplus \text{Map}_k(Z_i, G) \rightarrow \bigoplus \text{Map}_k(Z_{ij}, G)$$

is exact. \square

Unfortunately this result is only useful to understand $L_1\text{Alb}(X)$ for X “strictly reduced”, as we shall see below. In general, we shall have to consider Albanese schemes for the  h topology.

11.2. The  h topology. In this subsection and the next ones, we assume that k is of characteristic 0. Recall that $\text{HI}_{\text{ t}} = \text{HI}_{\text{ t}}^s$ in this case by Proposition D.1.4.

The following  tale analogue of the cdh topology was first considered by Thomas Geisser [36]:

11.2.1. Definition. The  h topology on $\text{Sch}(k)$ is the topology generated by the  tale topology and coverings defined by abstract blow-ups (it is the same as [86, Def. 4.1.9] by replacing the Nisnevich topology by the  tale topology).

As in [86, Th. 4.1.10] (see [61, Proof of Th. 14.20] for more details), one has:

11.2.2. Proposition. *Let $C \in \text{DM}_{-, \text{ t}}^{\text{eff}}$. Then, for any $X \in \text{Sch}(k)$ and any $q \in \mathbb{Z}$ one has*

$$\text{Hom}_{\text{DM}_{-, \text{ t}}^{\text{eff}}}(M_{\text{ t}}(X), C[q]) \simeq H_{\text{ h}}^q(X, C_{\text{ h}}[q]).$$

In particular, if X is smooth then $H_{\text{ t}}^q(X, C_{\text{ t}}) \xrightarrow{\sim} H_{\text{ h}}^q(X, C_{\text{ h}})$. \square

(See [36, Th. 4.3] for a different proof of the second statement.)

The following lemma will be used many times:

11.2.3. Lemma (Blow-up induction). *Let \mathcal{A} be an abelian category.*

a) Let $\mathcal{B} \subseteq \mathcal{A}$ be a thick subcategory and $H^ : \text{Sch}(k)^{\text{op}} \rightarrow \mathcal{A}^{(\mathbb{N})}$ a functor with the following property: given an abstract blow-up as in §8.1.e, we have a long exact sequence*

$$\cdots \rightarrow H^i(X) \rightarrow H^i(\tilde{X}) \oplus H^i(Z) \rightarrow H^i(\tilde{X}) \rightarrow H^{i+1}(X) \rightarrow \cdots$$

Let $n \geq 0$, and assume that $H^i(X) \in \mathcal{B}$ for $i \leq n$ and $X \in \text{Sm}(k)$. Then $H^i(X) \in \mathcal{B}$ for $i \leq n$ and all $X \in \text{Sch}(k)$.

b) Let H_1^, H_2^* be two functors as in a) and $\varphi^* : H_1^* \rightarrow H_2^*$ be a natural transformation. Let $n \geq 0$, and suppose that φ_X^i is an isomorphism for all $X \in \text{Sm}(k)$ and $i \leq n$. Then φ_X^i is an isomorphism for all $X \in \text{Sch}(k)$ and $i \leq n$.*

We get the same statements as a) and b) by replacing “ $i \leq n$ ” by “ $i \geq n + \dim X$ ”.

Proof. Induction on $\dim X$ in two steps: 1) if X is integral, choose a resolution of singularities $\tilde{X} \rightarrow X$; 2) in general, if Z_1, \dots, Z_r are the irreducible components of X , choose $\tilde{X} = \coprod Z_i$ and $Z = \bigcup_{i \neq j} Z_i \cap Z_j$. \square

11.2.4. Examples. 1) Thanks to [86, Th. 4.1.10] and Proposition 11.2.2, cdh or éh cohomology with coefficients in an object of $\text{DM}_{-,\text{ét}}^{\text{eff}}$ or $\text{DM}_{-,\text{ét}}^{\text{eff}}$ satisfy the hypothesis of a) (here $\mathcal{A} = \text{abelian groups}$). (See also [36, Prop. 3.2] for a different proof.)

2) Étale cohomology with torsion coefficients satisfies the hypothesis of a) by [69, Prop. 2.1] (recall that the proof of *loc. cit.* relies on the proper base change theorem).

Here is a variant of Lemma 11.2.3:

11.2.5. Proposition. *a) Let $X \in \text{Sch}(k)$ and $X_\bullet \rightarrow X$ be a hyper-envelope in the sense of Gillet-Soulé [38, 1.4.1]. Let $\tau = \text{cdh}$ or éh. Then, for any (bounded below) complex of sheaves C over $\text{Sch}(X)_\tau$, the augmentation map*

$$H_\tau^*(X, C) \rightarrow H_\tau^*(X_\bullet, C)$$

is an isomorphism.

b) Suppose that X_0 and X_1 are smooth and \mathcal{F} is a homotopy invariant Nisnevich (if $\tau = \text{cdh}$) or étale (if $\tau = \text{éh}$) sheaf with transfers. Then we have

$$\mathcal{F}_\tau(X) = \text{Ker}(\mathcal{F}(X_0) \rightarrow \mathcal{F}(X_1)).$$

Proof. a) By [61, Lemma 12.26], X_\bullet is a proper τ -hypercovering (cf. [18, p. 46]). Therefore the proposition follows from the standard theory of cohomological descent.

b) Let us take $C = \mathcal{F}_\tau[0]$. By a) we have a descent spectral sequence which gives a short exact sequence

$$0 \rightarrow \mathcal{F}_\tau(X) \rightarrow \mathcal{F}_\tau(X_0) \rightarrow \mathcal{F}_\tau(X_1)$$

and the conclusion now follows from 11.2.2. \square

11.3. $L_i\text{Alb}(X)$ for X singular. The following is a general method for computing the 1-motivic homology of $L\text{Alb}^\mathbb{Q}(X)$:

11.3.1. Proposition. *If $\text{char } k = 0$ and $X \in \text{Sch}(k)$ consider cdh cohomology groups $\mathbb{H}_{\text{cdh}}^i(X, \pi^*(N))_\mathbb{Q}$, where $\pi : \text{Sch}(k)_{\text{cdh}} \rightarrow \text{Sch}(k)_{\text{Zar}}$ is the canonical map from the cdh site to the big Zariski site. Then we have short exact sequences, for all $i \in \mathbb{Z}$*

$$\begin{aligned} 0 \rightarrow \text{Ext}^1(L_{i-1}\text{Alb}^\mathbb{Q}(X), N) &\rightarrow \mathbb{H}_{\text{cdh}}^i(X, \pi^*(N))_\mathbb{Q} \\ &\rightarrow \text{Hom}(L_i\text{Alb}^\mathbb{Q}(X), N) \rightarrow 0 \end{aligned}$$

$$\begin{aligned} 0 \rightarrow \text{Ext}^1(N, L_{i+1}\text{Alb}^\mathbb{Q}(X)) &\rightarrow \mathbb{E}\text{xt}^{-i}(N, L\text{Alb}^\mathbb{Q}(X)) \\ &\rightarrow \text{Hom}(N, L_i\text{Alb}(X)) \rightarrow 0. \end{aligned}$$

Proof. For any 1-motive $N \in \mathcal{M}_1$ we have a spectral sequence

$$(11.1) \quad E_2^{p,q} = \text{Ext}^p(L_q\text{Alb}(X), N) \implies \mathbb{E}\text{xt}^{p+q}(L\text{Alb}(X), N)$$

yielding the following short exact sequence

$$\begin{aligned} 0 \rightarrow \text{Ext}^1(L_{i-1}\text{Alb}^\mathbb{Q}(X), N) &\rightarrow \mathbb{E}\text{xt}^i(L\text{Alb}^\mathbb{Q}(X), N) \rightarrow \\ &\rightarrow \text{Hom}(L_i\text{Alb}^\mathbb{Q}(X), N) \rightarrow 0 \end{aligned}$$

because of Proposition 1.2.1. By adjunction we also obtain

$$\mathbb{E}\text{xt}^i(L\text{Alb}^\mathbb{Q}(X), N) = \text{Hom}(L\text{Alb}^\mathbb{Q}(X), N[i]) \cong \text{Hom}(M(X), \text{Tot } N[i]).$$

Now from [86, Thm. 3.2.6 and Cor. 3.2.7], for X smooth we have

$$\text{Hom}(M(X), \text{Tot } N[i]) \cong \mathbb{H}_{\text{Zar}}^i(X, N)_\mathbb{Q}.$$

If k is of characteristic 0 and X is arbitrary, we get the same isomorphism with cdh hypercohomology by [86, Thm. 4.1.10].

The proof for the second short exact sequence is similar. \square

The following proposition follows readily by blow-up induction (Lemma 11.2.3) from Corollary 10.2.3 and the exact sequences (8.2):

11.3.2. Proposition. *For any $X \in \text{Sch}(k)$ of dimension d in characteristic 0, we have*

- a) $L_i\text{Alb}(X) = 0$ if $i < 0$.
- b) $L_0\text{Alb}(X) = [\mathbb{Z}[\pi_0(X)] \rightarrow 0]$.

c) $L_i \text{Alb}(X) = 0$ for $i > \max(2, d+1)$.

d) $L_{d+1} \text{Alb}(X)$ is a group of multiplicative type. \square

11.4. The cohomological 1-motives $R^i \text{Pic}(X)$. If $X \in \text{Sch}(k)$, we quote the following variant of Proposition 11.3.1:

11.4.1. Lemma. *Let $N \in \mathcal{M}_1 \otimes \mathbb{Q}$ and $X \in \text{Sch}(k)$. We have a short exact sequence, for all $i \in \mathbb{Z}$*

$$0 \rightarrow \text{Ext}(N, R^{i-1} \text{Pic}(X)) \rightarrow \mathbb{H}_{\text{cdh}}^i(X, \pi^*(N^*))_{\mathbb{Q}} \rightarrow \text{Hom}(N, R^i \text{Pic}(X)) \rightarrow 0$$

here $\pi : \text{Sch}(k)_{\text{cdh}} \rightarrow \text{Sch}(k)_{\text{zar}}$ and N^* is the Cartier dual.

Proof. The spectral sequence

$$E_2^{p,q} = \text{Ext}^p(N, R^q \text{Pic}(X)) \implies \mathbb{E} \text{Ext}^{p+q}(N, R \text{Pic}(X))$$

yields the following short exact sequence

$$0 \rightarrow \text{Ext}(N, R^{i-1} \text{Pic}(X)) \rightarrow \mathbb{E} \text{Ext}^i(N, R \text{Pic}(X)) \rightarrow \text{Hom}(N, R^i \text{Pic}(X)) \rightarrow 0$$

and by Cartier duality, the universal property and [86, Thm. 4.1.10] we obtain:

$$\begin{aligned} \mathbb{E} \text{Ext}^i(N, R \text{Pic}(X)) &= \text{Hom}(N, R \text{Pic}(X)[i]) \cong \text{Hom}(L \text{Alb}(X), N^*[i]) = \\ &= \text{Hom}(M(X), N^*[i]) \cong \mathbb{H}_{\text{cdh}}^i(X, \pi^*(N^*))_{\mathbb{Q}}. \end{aligned}$$

\square

On the other hand, here is a dual to Proposition 11.3.2:

11.4.2. Proposition. *For any $X \in \text{Sch}(k)$ of dimension d in characteristic 0, we have*

a) $R^i \text{Pic}(X) = 0$ if $i < 0$.

b) $R^0 \text{Pic}(X) = [0 \rightarrow \mathbb{Z}[\pi_0(X)]^*]$.

c) $R^i \text{Pic}(X) = 0$ for $i > \max(2, d+1)$.

d) $R^{d+1} \text{Pic}(X)$ is discrete. \square

11.5. Borel-Moore variants.

11.5.1. Definition. For $X \in \text{Sch}(k)$, we denote by $\pi_0^c(X)$ the disjoint union of $\pi_0(Z_i)$ where Z_i runs through the *proper* connected components of X : this is the *proper scheme of constants*.

11.5.2. Proposition. *Let $X \in \text{Sch}(k)$ of dimension d . Then:*

a) $L_i \text{Alb}^c(X) = 0$ if $i < 0$.

b) $L_0 \text{Alb}^c(X) = [\mathbb{Z}[\pi_0^c(X)] \rightarrow 0]$. In particular, $L_0 \text{Alb}^c(X) = 0$ if no

connected component is proper.

c) $L_i \mathrm{Alb}^c(X) = 0$ for $i > \max(2, d+1)$.

d) $L_{d+1} \mathrm{Alb}^c(X)$ is a group of multiplicative type.

Proof. If X is proper, this is Proposition 11.3.2. In general, we may choose a compactification \overline{X} of X ; if $Z = \overline{X} - X$, with $\dim Z < \dim X$, the claim follows inductively by the long exact sequence (9.1). \square

We leave it to the reader to formulate the dual of this proposition for $\mathrm{RPic}^c(X)$.

12. 1-MOTIVIC HOMOLOGY AND COHOMOLOGY OF CURVES

12.1. “Chow-Künneth” decomposition for a curve. Note that for any curve C , the map a_C is an isomorphism by Proposition 5.1.4. Moreover, since the category of 1-motives up to isogeny is of cohomological dimension 1 (see Prop. 1.2.1), the complex $L\mathrm{Alb}^{\mathbb{Q}}(C)$ can be represented by a complex with zero differentials. Using Proposition 11.3.2 c), we then have:

12.1.1. Corollary. *If C is a curve then the motive $M(C)$ decomposes in $\mathrm{DM}_{\mathrm{gm}}^{\mathrm{eff}} \otimes \mathbb{Q}$ as*

$$M(C) = M_0(C) \oplus M_1(C) \oplus M_2(C)$$

where $M_i(C) := \mathrm{Tot} L_i \mathrm{Alb}^{\mathbb{Q}}(C)[i]$.

12.2. $L_i \mathrm{Alb}$ and $R^i \mathrm{Pic}$ of curves. Here we shall complete the computation of Proposition 11.3.2 in the case of a curve C .

Let \tilde{C} denote the normalisation of C . Let \overline{C} be a smooth compactification of \tilde{C} so that $F = \overline{C} - \tilde{C}$ is a finite set of closed points. Consider the following cartesian square

$$\begin{array}{ccc} \tilde{S} & \longrightarrow & \tilde{C} \\ \downarrow & & \downarrow \\ S & \longrightarrow & C \end{array}$$

where S denote the singular locus. Let \overline{S} denote \tilde{S} regarded in \overline{C} . Note that $S = \pi_0(S)$, $\tilde{S} = \pi_0(\tilde{S})$ and $\pi_0(\tilde{S}) \xrightarrow{\cong} \pi_0(S)$ if $\tilde{C} \rightarrow C$ is radicial, yielding $M(\tilde{C}) \xrightarrow{\cong} M(C)$ in this case. In general, we have the following.

12.2.1. **Theorem.** *Let $C, \tilde{C}, \overline{C}, S, \tilde{S}, \overline{S}$ and F as above. Then*

$$L_i \text{Alb}(C) = \begin{cases} [\mathbb{Z}[\pi_0(C)] \rightarrow 0] & \text{if } i = 0 \\ [\text{Div}_{\tilde{S}/S}^0(\overline{C}, F) \xrightarrow{u} \text{Pic}^0(\overline{C}, F)] & \text{if } i = 1 \\ [0 \rightarrow \text{NS}_{\tilde{C}/k}^*] & \text{if } i = 2 \\ 0 & \text{otherwise} \end{cases}$$

where: $\text{Div}_{\tilde{S}/S}^0(\overline{C}, F) = \text{Div}_{\tilde{S}/S}^0(\tilde{C})$ here is the free group of degree zero divisors generated by \tilde{S} having trivial push-forward on S and the map u is the canonical map (cf. [8, Def. 2.2.1]); $\text{NS}_{\tilde{C}/k}$ is the sheaf associated to the free abelian group on the proper irreducible components of \tilde{C} . In particular, $L_1 \text{Alb}(C) = \text{Pic}^-(C)$.

Proof. We use the long exact sequence (8.2)

$$\cdots \rightarrow L_i \text{Alb}(\tilde{S}) \rightarrow L_i \text{Alb}(\tilde{C}) \oplus L_i \text{Alb}(S) \rightarrow L_i \text{Alb}(C) \rightarrow L_{i-1} \text{Alb}(\tilde{S}) \rightarrow \cdots$$

Since S and \tilde{S} are 0-dimensional we have $L_i \text{Alb}(\tilde{S}) = L_i \text{Alb}(S) = 0$ for $i > 0$, therefore

$$L_i \text{Alb}(C) = L_i \text{Alb}(\tilde{C}) \text{ for } i \geq 2$$

and by 10.2.3 we get the claimed vanishing and description of $L_2 \text{Alb}(C)$. For $i = 0$ see Corollary 11.3.2. If $i = 1$ then $L_1 \text{Alb}(C)$ is here represented as an element of $\text{Ext}([\Lambda \rightarrow 0], L_1 \text{Alb}(\tilde{C}))$ where $\Lambda := \text{Ker}(\mathbb{Z}[\pi_0(\tilde{S})] \rightarrow \mathbb{Z}[\pi_0(\tilde{C})] \oplus \mathbb{Z}[\pi_0(S)])$. Recall, see 10.2.3, that $L_1 \text{Alb}(\tilde{C}) = [0 \rightarrow \mathcal{A}_{\tilde{C}/k}^0]$ thus $\text{Ext}(\Lambda, L_1 \text{Alb}(\tilde{C})) = \text{Hom}_k(\Lambda, \mathcal{A}_{\tilde{C}/k}^0)$ and

$$L_1 \text{Alb}(C) = [\Lambda \xrightarrow{u} \mathcal{A}_{\tilde{C}/k}^0].$$

Now $\Lambda = \text{Div}_{\tilde{S}/S}^0(\overline{C}, F)$, $\mathcal{A}_{\tilde{C}/k}^0 = \text{Pic}^0(\overline{C}, F)$ and the map u is induced by the following canonical map.

Consider $\varphi_{\tilde{C}} : \tilde{C} \rightarrow \text{Pic}(\overline{C}, F)$ where $\varphi_{\tilde{C}}(P) := (\mathcal{O}_{\overline{C}}(P), 1)$ yielding $\mathcal{A}_{\tilde{C}/k} = \text{Pic}(\overline{C}, F)$ and such that

$$0 \rightarrow \text{Div}_F^0(\overline{C})^* \rightarrow \text{Pic}^0(\overline{C}, F) \rightarrow \text{Pic}^0(\overline{C}) \rightarrow 0.$$

Thus $L_1 \text{Alb}(\tilde{C}) = [0 \rightarrow \text{Pic}^0(\overline{C}, F)]$. Note that $\mathbb{Z}[\pi_0(\tilde{S})] = \text{Div}_{\tilde{S}}(\tilde{C}) = \text{Div}_{\tilde{S}}(\overline{C}, F)$, the map $\mathbb{Z}[\pi_0(\tilde{S})] \rightarrow \mathbb{Z}[\pi_0(\tilde{C})]$ is the degree map and the following map $\mathbb{Z}[\pi_0(\tilde{S})] \rightarrow \mathbb{Z}[\pi_0(S)]$ is the proper push-forward of Weil divisors, i.e. $\Lambda = \text{Div}_{\tilde{S}/S}^0(\overline{C}, F)$. The map $\varphi_{\tilde{C}}$ then induces the mapping $u \in \text{Hom}_k(\Lambda, \text{Pic}^0(\overline{C}, F))$ which also is the canonical lifting of the universal map $D \mapsto \mathcal{O}_{\overline{C}}(D)$ as the support of D is disjoint from F (cf. [8, Lemma 3.1.3]). \square

12.2.2. Remark. Note that $R^1\text{Pic}(C) = L_1\text{Alb}(C)^*$ is Deligne's motivic cohomology $H_m^1(C)(1)$ of the singular curve C by [8, Prop. 3.1.2]. In fact, $\text{Pic}^-(C)^* = \text{Alb}^+(C) \cong H_m^1(C)(1) \cong \text{Pic}^+(C) = \text{Alb}^-(C)^*$ for a curve C . Thus $L_1\text{Alb}(C)$ also coincides with the homological Albanese 1-motive $\text{Alb}^-(C)$. The $L_i\text{Alb}(C)$ also coincide with Lichtenbaum-Deligne motivic homology $h_i(C)$ of the curve C , cf. [55].

12.2.3. Corollary. *Let C be a curve, C' its seminormalisation, \overline{C}' a compactification of C' , and $F = \overline{C}' - C'$. Let further \tilde{C} denote the normalisation of C . Then*

$$R^i\text{Pic}(C) = \begin{cases} [0 \rightarrow \mathbb{G}_m[\pi_0(C)]] & \text{if } i = 0 \\ [\text{Div}_F^0(\overline{C}') \rightarrow \text{Pic}^0(\overline{C}')] & \text{if } i = 1 \\ [\text{NS}(\tilde{C}) \rightarrow 0] & \text{if } i = 2 \\ 0 & \text{otherwise} \end{cases}$$

where $\text{NS}(\tilde{C}) = \mathbb{Z}[\pi_0^c(\tilde{C})]$ and $\pi_0^c(\tilde{C})$ is the scheme of proper constants. In particular, $R^1\text{Pic}(C) \cong \text{Pic}^+(C)$.

12.3. Borel-Moore variants.

12.3.1. Theorem. *Let C be a smooth curve, \overline{C} a smooth compactification of C and $F = \overline{C} - C$ the finite set of closed points at infinity. Then*

$$L_i\text{Alb}^c(C) = \begin{cases} [\mathbb{Z}[\pi_0^c(C)] \rightarrow 0] & \text{if } i = 0 \\ [\text{Div}_F^0(\overline{C}) \rightarrow \text{Pic}^0(\overline{C})] & \text{if } i = 1 \\ [0 \rightarrow \text{NS}_{\overline{C}/k}^*] & \text{if } i = 2 \\ 0 & \text{otherwise} \end{cases}$$

where $\text{NS}(\overline{C}) = \mathbb{Z}[\pi_0(\overline{C})]$ and $\pi_0^c(C)$ is the scheme of proper constants.

Proof. It follows from the distinguished triangle

$$\begin{array}{ccc} L\text{Alb}(F) & \longrightarrow & L\text{Alb}(\overline{C}) \\ +1 \nearrow & & \swarrow \\ & L\text{Alb}^c(C) & \end{array}$$

and Corollary 10.2.3, yielding the claimed description: $L_0\text{Alb}^c(C) = \text{Coker}(L_0\text{Alb}(F) \rightarrow L_0\text{Alb}(\overline{C}))$ moreover we have

$$[\text{Div}_F^0(\overline{C}) \rightarrow 0] = \text{Ker}(L_0\text{Alb}(F) \rightarrow L_0\text{Alb}(\overline{C}))$$

and the following extension

$$0 \rightarrow L_1\text{Alb}(\overline{C}) \rightarrow L_1\text{Alb}^c(C) \rightarrow [\text{Div}_F^0(\overline{C}) \rightarrow 0] \rightarrow 0.$$

Finally, $L_i\text{Alb}(\overline{C}) = L_i\text{Alb}^c(C)$ for $i \geq 2$. \square

12.3.2. Corollary. *Let C be a smooth curve, \overline{C} a smooth compactification of C and $F = \overline{C} - C$ the finite set of closed points at infinity. Then*

$$R^i \text{Pic}^c(C) = \begin{cases} [0 \rightarrow \mathbb{G}_m[\pi_0^c(C)]] & \text{if } i = 0 \\ [0 \rightarrow \text{Pic}^0(\overline{C}, F)] & \text{if } i = 1 \\ [\text{NS}(\overline{C}) \rightarrow 0] & \text{if } i = 2 \\ 0 & \text{otherwise} \end{cases}$$

where $\text{NS}(\overline{C}) = \mathbb{Z}[\pi_0(\overline{C})]$ and $\pi_0^c(C)$ is the scheme of proper constants.

Here we have that $R^1 \text{Pic}^c(C) = R^1 \text{Pic}^*(C)$ is also the Albanese variety of the smooth curve.

Note that $L_1 \text{Alb}^c(C)$ ($= \text{Pic}^+(C) = \text{Alb}^+(C)$ for curves, see [8]) coincide with Deligne's motivic $H_m^1(C)(1)$ of the smooth curve C . This is due to the Poincaré duality isomorphism $M^c(C) = M(C)^*(1)[2]$.

13. COMPARISON WITH Pic^+ , Pic^- , Alb^+ AND Alb^-

In this section, k is of characteristic 0 since we mainly deal with singular schemes. We want to study $L_1 \text{Alb}(X)$ and its variants in more detail. In particular, we show in Proposition 13.6.3 c) that it is always a Deligne 1-motive, and show in Corollaries 13.7.2 and 13.10.2 that, if X is normal or proper, it is canonically isomorphic to the 1-motive $\text{Alb}^-(X)$ of [8]. Precise descriptions of $L_1 \text{Alb}(X)$ are given in Proposition 13.6.5 and Corollary 13.6.6.

We also describe $L_1 \text{Alb}^c(X)$ in Proposition 13.11.2; more precisely, we prove in Theorem 13.11.1 that its dual $R^1 \text{Pic}^c(X)$ is canonically isomorphic to $\text{Pic}^0(\overline{X}, Z)/\mathcal{U}$, where \overline{X} is a compactification of X with complement Z and \mathcal{U} is the unipotent radical of the commutative algebraic group $\text{Pic}^0(\overline{X}, Z)$. Finally, we prove in Theorem 13.12.6 that $L_1 \text{Alb}^*(X)$ is abstractly isomorphic to the 1-motive $\text{Alb}^+(X)$ of [8].

We start with some comparison results between éh and étale cohomology for non smooth schemes.

Let $\varepsilon : \text{Sch}_{\text{éh}} \rightarrow \text{Sch}_{\text{ét}}$ be the obvious morphism of sites. If \mathcal{F} is an étale sheaf on Sch , we denote by $\mathcal{F}_{\text{éh}}$ its éh sheafification (that is, $\mathcal{F}_{\text{éh}} = \varepsilon_* \varepsilon^* \mathcal{F}$). We shall abbreviate $H_{\text{éh}}^*(X, \mathcal{F}_{\text{éh}})$ to $H_{\text{éh}}^*(X, \mathcal{F})$.

13.1. Torsion sheaves. The first basic result is a variant of [83, Cor. 7.8 and Th. 10.7]: it follows from Proposition 11.2.2 and Examples 11.2.4 via Lemma 11.2.3 b).

13.1.1. Proposition. *Let C be a bounded below complex of torsion sheaves on $(\text{Spec } k)_{\text{ét}}$. Then, for any $X \in \text{Sch}$ and any $n \in \mathbb{Z}$, $H_{\text{éh}}^n(X, C) \xrightarrow{\sim} H_{\text{ét}}^n(X, C)$. \square*

(See [36, Th. 3.6] for a different proof.)

13.2. Discrete sheaves.

13.2.1. Lemma. *If \mathcal{F} is discrete, then*

a) $\mathcal{F} \xrightarrow{\sim} \mathcal{F}_{\text{éh}}$. *More precisely, for any $X \in \text{Sch}$, $\mathcal{F}(\pi_0(X)) \xrightarrow{\sim} \mathcal{F}(X) \xrightarrow{\sim} \mathcal{F}_{\text{éh}}(X)$.*

b) *If $f : Y \rightarrow X$ is surjective with geometrically connected fibres, then $\mathcal{F}_{\text{éh}} \xrightarrow{\sim} f_* f^* \mathcal{F}_{\text{éh}}$.*

Proof. a) We may assume X reduced. Clearly it suffices to prove that $\mathcal{F}(\pi_0(X)) \xrightarrow{\sim} \mathcal{F}_{\text{éh}}(X)$ for any $X \in \text{Sch}$. In the situation of §8.1.e, we have a commutative diagram of exact sequences

$$\begin{array}{ccccccc} 0 & \longrightarrow & \mathcal{F}_{\text{éh}}(X) & \longrightarrow & \mathcal{F}_{\text{éh}}(\tilde{X}) \oplus \mathcal{F}_{\text{éh}}(Z) & \longrightarrow & \mathcal{F}_{\text{éh}}(\tilde{Z}) \\ & & \uparrow & & \uparrow & & \uparrow \\ 0 & \longrightarrow & \mathcal{F}(\pi_0(X)) & \longrightarrow & \mathcal{F}(\pi_0(\tilde{X})) \oplus \mathcal{F}(\pi_0(Z)) & \longrightarrow & \mathcal{F}(\pi_0(\tilde{Z})). \end{array}$$

The proof then goes exactly as the one of Proposition 13.1.1. b) follows from a). \square

It is well-known that $H_{\text{ét}}^1(X, \mathcal{F}) = 0$ for any geometrically unibranch scheme $X \in \text{Sch}$ if \mathcal{F} is constant and torsion-free (cf. [SGA4, IX, Prop. 3.6 (ii)]). The following lemma shows that this is also true for the éh topology, at least if X is normal.

13.2.2. Lemma (compare [61, Ex. 12.31 and 12.32]). *Let \mathcal{F} be a constant torsion-free sheaf on $\text{Sch}(k)$.*

a) *For any $X \in \text{Sch}$, $H_{\text{éh}}^1(X, \mathcal{F})$ is torsion-free. It is finitely generated if \mathcal{F} is a lattice.*

b) *Let $f : \tilde{X} \rightarrow X$ be a surjective morphism. Then $H_{\text{éh}}^1(X, \mathcal{F}) \rightarrow H_{\text{éh}}^1(\tilde{X}, \mathcal{F})$ is injective in the following cases:*

- (i) *The geometric fibres of f are connected.*
- (ii) *f is finite and flat.*

c) *If X is normal, $H_{\text{éh}}^1(X, \mathcal{F}) = 0$.*

Proof. a) The first assertion follows immediately from Lemma 13.2.1 (consider the exact sequence of multiplication by n on \mathcal{F}). The second assertion follows by blow-up induction from the fact that $H_{\text{éh}}^1(X, \mathcal{F}) = 0$ if X is smooth, by Proposition 11.2.2.

b) In the first case, the Leray spectral sequence yields an injection

$$H_{\text{éh}}^1(X, f_* \mathcal{F}) \hookrightarrow H_{\text{éh}}^1(\tilde{X}, \mathcal{F})$$

and $f_*\mathcal{F} = \mathcal{F}$ by Lemma 13.2.1 b). In the second case, the theory of trace [SGA4, XVII, Th. 6.2.3] provides \mathcal{F} , hence $\mathcal{F}_{\text{éh}}$, with a morphism $\text{Tr}_f : f_*\mathcal{F} \rightarrow \mathcal{F}$ whose composition with the natural morphism is (on each connected component of X) multiplication by some nonzero integer. This shows that the kernel of $H_{\text{éh}}^1(X, \mathcal{F}) \rightarrow H_{\text{éh}}^1(X, f_*\mathcal{F})$ is torsion, hence 0 by a).

c) follows from b) with \tilde{X} a desingularisation of X : by Proposition 13.1.1, $H_{\text{ét}}^1(\tilde{X}, \mathcal{F}) \xrightarrow{\sim} H_{\text{éh}}^1(\tilde{X}, \mathcal{F})$ and it is well-known that the first group is 0; on the other hand, the fibres of f are geometrically connected by Zariski's main theorem. \square

The following is a version of [90, Lemma 5.6]:

13.2.3. Lemma. *Let $f : \tilde{X} \rightarrow X$ be a finite birational morphism, $i : Z \hookrightarrow X$ a closed subset and $\tilde{Z} = p^{-1}(Z)$. Then, for any discrete sheaf, we have a long exact sequence:*

$$\begin{aligned} \cdots \rightarrow H_{\text{ét}}^i(X, \mathcal{F}) &\rightarrow H_{\text{ét}}^i(\tilde{X}, \mathcal{F}) \oplus H_{\text{ét}}^i(Z, \mathcal{F}) \\ &\rightarrow H_{\text{ét}}^i(\tilde{Z}, \mathcal{F}) \rightarrow H_{\text{ét}}^{i+1}(X, \mathcal{F}) \rightarrow \cdots \end{aligned}$$

Proof. Let $g : \tilde{Z} \rightarrow Z$ be the induced map. Then f_*, i_* and g_* are exact for the étale topology. Thus it suffices to show that the sequence of sheaves

$$0 \rightarrow \mathcal{F} \rightarrow f_*f^*\mathcal{F} \oplus i_*i^*\mathcal{F} \rightarrow (ig)_*(ig)^*\mathcal{F} \rightarrow 0$$

is exact. The assertion is local for the étale topology, hence we may assume that X is strictly local. Then Z, \tilde{X} and \tilde{Z} are strictly local as well, hence connected, thus the statement is obvious. \square

We can now prove:

13.2.4. Proposition. *For any $X \in \text{Sch}(k)$ and any discrete sheaf \mathcal{F} , the map $H_{\text{ét}}^1(X, \mathcal{F}) \rightarrow H_{\text{éh}}^1(X, \mathcal{F})$ is an isomorphism.*

Proof. Consider the exact sequence

$$0 \rightarrow \mathcal{F}_{\text{tors}} \rightarrow \mathcal{F} \rightarrow \mathcal{F}/\mathcal{F}_{\text{tors}} \rightarrow 0$$

where $\mathcal{F}_{\text{tors}}$ is the torsion subsheaf of \mathcal{F} . The 5 lemma, Lemma 13.2.1 and Proposition 13.1.1 reduce us to the case where \mathcal{F} is torsion-free. As a discrete étale sheaf over $\text{Sch}(k)$, \mathcal{F} becomes constant over some finite Galois extension of k : a Hochschild-Serre argument then reduces us to the case where \mathcal{F} is *constant and torsion-free*.

Let $f : \tilde{X} \rightarrow X$ be the normalisation of X , and take for Z the non-normal locus of X in Lemma 13.2.3. The result now follows from

comparing the exact sequence of this lemma with the one for éh topology, and using Lemma 13.2.2 c). \square

13.2.5. Corollary. *The exact sequence of Lemma 13.2.3 holds up to $i = 1$ for a general abstract blow-up.* \square

13.3. Strictly reduced schemes. If G is a commutative k -group scheme, the associated presheaf \underline{G} is an étale sheaf on reduced k -schemes of finite type. However, $\underline{G}(X) \rightarrow \underline{G}_{\text{éh}}(X)$ is not an isomorphism in general if X is not smooth. Nevertheless we have some nice results in Lemma 13.3.4 below.

13.3.1. Definition. A separated k -scheme of finite type X is *strictly reduced* (a recursive definition) if it is reduced and

- (i) If X is irreducible: X_{sing} , considered with its reduced structure, is strictly reduced.
- (ii) If Z_1, \dots, Z_n are the irreducible components of X : all Z_i are strictly reduced and the scheme-theoretic intersection $Z_i \cap Z_j$ is reduced for any $i \neq j$.

13.3.2. Examples. 1) If $\dim X = 0$, X is strictly reduced.
 2) The union of a line and a tangent parabola is not strictly reduced.
 3) If X is normal and of dimension ≤ 2 , it is strictly reduced.
 4) M. Ojanguren provided the following example of a normal 3-fold which is not strictly reduced: take the affine hypersurface with equation $uv = x^2(y^2 - x)^2$.

13.3.3. Lemma. *Let G be an affine group scheme and $f : Y \rightarrow X$ a proper surjective map with geometrically connected fibres. Then $\underline{G}(X) \xrightarrow{\sim} \underline{G}(Y)$, and $H_\tau^0(X, \underline{G}) \xrightarrow{\sim} H_\tau^0(Y, \underline{G})$ for any Grothendieck topology τ stronger than the Zariski topology.*

Proof. The first statement is clear, and the second follows because the hypothesis on f is stable under any base change. \square

13.3.4. Lemma. *a) If X is reduced, then the map*

$$(13.1) \quad \underline{G}(X) \rightarrow \underline{G}_{\text{éh}}(X)$$

is injective for any semi-abelian k -scheme G .

b) If X is strictly reduced, (13.1) is an isomorphism.

c) If X is proper and G is a torus, the maps $\underline{G}(\pi_0(X)) \rightarrow \underline{G}_{\text{ét}}(X) \rightarrow \underline{G}_{\text{éh}}(X)$ are isomorphisms. If moreover X is reduced, (13.1) is an isomorphism.

Proof. a) Let Z_i be the irreducible components of X (with their reduced structure), and for each i let $p_i : \tilde{Z}_i \rightarrow Z_i$ be a resolution of singularities.

We have a commutative diagram

$$\begin{array}{ccccc}
 \underline{G}_{\text{éh}}(X) & \longrightarrow & \bigoplus \underline{G}_{\text{éh}}(Z_i) & \xrightarrow{(p_i^*)} & \bigoplus \underline{G}_{\text{éh}}(\tilde{Z}_i) \\
 \uparrow & & \uparrow & & \uparrow \\
 \underline{G}(X) & \longrightarrow & \bigoplus \underline{G}(Z_i) & \xrightarrow{(p_i^*)} & \bigoplus \underline{G}(\tilde{Z}_i).
 \end{array}$$

The bottom horizontal maps are injective; the right vertical map is an isomorphism by Proposition 11.2.2. The claim follows.

b) We argue by induction on $d = \dim X$. If $d = 0$ this is trivial. If $d > 0$, we first assume X irreducible. Let Z be its singular locus, and choose a desingularisation $p : \tilde{X} \rightarrow X$ with p proper surjective, \tilde{X} smooth, $\tilde{Z} = p^{-1}(Z)$ a divisor with normal crossings (in particular reduced) and $p|_{\tilde{X}-\tilde{Z}}$ an isomorphism. We now have a commutative diagram

$$\begin{array}{ccccccc}
 0 & \longrightarrow & \underline{G}_{\text{éh}}(X) & \longrightarrow & \underline{G}_{\text{éh}}(Z) \oplus \underline{G}_{\text{éh}}(\tilde{X}) & \longrightarrow & \bigoplus \underline{G}_{\text{éh}}(\tilde{Z}) \\
 & & \uparrow & & \uparrow & & \uparrow \\
 0 & \longrightarrow & \underline{G}(X) & \longrightarrow & \underline{G}(Z) \oplus \underline{G}(\tilde{X}) & \longrightarrow & \bigoplus \underline{G}(\tilde{Z})
 \end{array}$$

where the lower sequence is exact, the middle vertical map is bijective by induction on d and the smooth case (Proposition 11.2.2) and the right vertical map is injective by a). It follows that the left vertical map is surjective.

In general, write Z_1, \dots, Z_n for the irreducible components of X : by assumption, the two-fold intersections Z_{ij} are reduced. The commutative diagram

$$\begin{array}{ccccccc}
 0 & \longrightarrow & \underline{G}_{\text{éh}}(X) & \longrightarrow & \bigoplus \underline{G}_{\text{éh}}(Z_i) & \longrightarrow & \bigoplus \underline{G}_{\text{éh}}(Z_{ij}) \\
 & & \uparrow & & \uparrow & & \uparrow \\
 0 & \longrightarrow & \underline{G}(X) & \longrightarrow & \bigoplus \underline{G}(Z_i) & \longrightarrow & \bigoplus \underline{G}(Z_{ij})
 \end{array}$$

then has the same formal properties as the previous one, and we conclude.

For c), same proof as for Lemma 13.2.1 a). (The second statement of c) is true because $\underline{G}(\pi_0(X)) \xrightarrow{\sim} \underline{G}(X)$ if X is proper and reduced.) \square

13.4. Normal schemes. The main result of this subsection is:

13.4.1. Theorem. *Let X be normal. Then, for any $\mathcal{F} \in \text{Shv}_1$, the map $\mathcal{F}(X) \rightarrow \mathcal{F}_{\text{éh}}(X)$ is bijective and the map $H_{\text{ét}}^1(X, \mathcal{F}) \rightarrow H_{\text{éh}}^1(X, \mathcal{F}_{\text{éh}})$ is injective (with torsion-free cokernel by Proposition 13.1.1).*

Proof. In several steps:

Step 1. The first result implies the second for a given sheaf \mathcal{F} : let $\varepsilon : \text{Sch}(k)_{\text{éh}} \rightarrow \text{Sch}(k)_{\text{ét}}$ be the projection morphism. The associated Leray spectral sequence gives an injection

$$H_{\text{ét}}^1(X, \varepsilon_* \mathcal{F}_{\text{éh}}) \hookrightarrow H_{\text{éh}}^1(X, \mathcal{F}_{\text{éh}}).$$

But any scheme étale over X is normal [SGA1, Exp. I, Cor. 9.10], therefore $\mathcal{F} \rightarrow \varepsilon_* \mathcal{F}_{\text{éh}}$ is an isomorphism over the small étale site.

Step 2. Let $0 \rightarrow \mathcal{F}' \rightarrow \mathcal{F} \rightarrow \mathcal{F}'' \rightarrow 0$ be a short exact sequence in Shv_1 . If the theorem is true for \mathcal{F}' and \mathcal{F}'' , it is true for \mathcal{F} . This follows readily from *Step 1* and a diagram chase.

Step 3. Given the structure of 1-motivic sheaves, *Step 1 - 2* reduce us to prove that $\mathcal{F}(X) \xrightarrow{\sim} \mathcal{F}_{\text{éh}}(X)$ separately when \mathcal{F} is discrete, a torus or an abelian variety. The discrete case follows from Lemma 13.2.1 a).

Step 4. If G is a torus, let $\pi : \tilde{X} \rightarrow X$ be a desingularisation of X . We have a commutative diagram

$$\begin{array}{ccc} \underline{G}_{\text{éh}}(X) & \longrightarrow & \underline{G}_{\text{éh}}(\tilde{X}) \\ \uparrow & & \uparrow \\ \underline{G}(X) & \longrightarrow & \underline{G}(\tilde{X}). \end{array}$$

Here the right vertical map is an isomorphism because \tilde{X} is smooth and the two horizontal maps are also isomorphisms by Lemma 13.3.3 applied to π (Zariski's main theorem). The result follows.

Step 5. Let finally G be an abelian variety. This time, it is not true in general that $\underline{G}(X) \xrightarrow{\sim} \underline{G}(\tilde{X})$ for a smooth desingularisation \tilde{X} of X . However, we get the result from Proposition 11.2.5 b) and the following general lemma. \square

13.4.2. Lemma. *Let $X \in \text{Sch}(k)$ be normal, $p : X_0 \rightarrow X$ a proper surjective map such that the restriction of p to a suitable connected component X'_0 of X_0 is birational. Let $X_1 \xrightarrow[p_1]{p_0} X_0$ be two morphisms such that $pp_0 = pp_1$ and that the induced map $\Psi : X_1 \rightarrow X_0 \times_X X_0$ is proper surjective. Let $Y \in \text{Sch}(k)$ and let $f : X_0 \rightarrow Y$ be such that $fp_0 = fp_1$.*

$$\begin{array}{ccc} X_1 & & \\ p_0 \downarrow & p_1 \downarrow & \\ X_0 & & \\ p \downarrow & f \searrow & \\ X & & Y \end{array}$$

Then there exists a unique morphism $\bar{f} : X \rightarrow Y$ such that $f = \bar{f}p$.

Proof. We may assume X connected. Since Ψ is proper surjective, the hypothesis is true by replacing X_1 by $X_0 \times_X X_0$, which we shall assume henceforth. Let $x \in X$ and $K = k(x)$. Base-changing by the morphism $\text{Spec } K \rightarrow X$, we find (by faithful flatness) that f is constant on $p^{-1}(x)$. Since p is surjective, this defines \bar{f} as a set-theoretic map, and this map is continuous for the Zariski topology because p is also proper.

It remains to show that \bar{f} is a map of locally ringed spaces. Let $x \in X$, $y = \bar{f}(x)$ and $x' \in p^{-1}(x) \cap X'_0$. Then $f^\#(\mathcal{O}_{Y,y}) \subseteq \mathcal{O}_{X'_0,x'}$. Note that X and X'_0 have the same function field L , and $\mathcal{O}_{X,x} \subseteq \mathcal{O}_{X'_0,x'} \subseteq L$. Now, since X is normal, $\mathcal{O}_{X,x}$ is the intersection of the valuation rings containing it.

Let \mathcal{O} be such a valuation ring, so that x is the centre of \mathcal{O} on X . By the valuative criterion of properness, we may find $x' \in p^{-1}(x) \cap X'_0$ such that $\mathcal{O}_{X'_0,x'} \subseteq \mathcal{O}$. This shows that

$$\mathcal{O}_{X,x} = \bigcap_{x' \in p^{-1}(x) \cap X'_0} \mathcal{O}_{X'_0,x'}$$

and therefore that $f^\#(\mathcal{O}_{Y,y}) \subseteq \mathcal{O}_{X,x}$. Moreover, the corresponding map $\bar{f}^\# : \mathcal{O}_{Y,y} \rightarrow \mathcal{O}_{X,x}$ is local since $f^\#$ is.

(Alternatively, observe that f and its topological factorisation induce a map

$$f^\# : \mathcal{O}_Y \rightarrow f_* \mathcal{O}_{X_0} \simeq \bar{f}_* p_* \mathcal{O}_{X_0} = \bar{f}_* \mathcal{O}_X.)$$

□

13.5. Some representability results.

13.5.1. Proposition. *Let π^X be the structural morphism of X and $(\pi^X_*)^{\text{éh}}$ the induced direct image morphism on the éh sites. For any $\mathcal{F} \in \text{HI}_{\text{ét}}$, let us denote the restriction of $R^q(\pi^X_*)^{\text{éh}} \mathcal{F}_{\text{éh}}$ to Sm by $\underline{R}^q \pi^X_* \mathcal{F}$ (in other words, $R^q(\pi^X_*)^{\text{éh}} \mathcal{F}_{\text{éh}}$ is the sheaf on $\text{Sm}(k)_{\text{ét}}$ associated to the presheaf $U \mapsto H^q_{\text{éh}}(X \times U, \mathcal{F}_{\text{éh}})$): it is an object of $\text{HI}_{\text{ét}}$. Then*

- a) *For any lattice L , $\underline{R}^q \pi^X_* L$ is a ind-discrete sheaf for all $q \geq 0$; it is a lattice for $q = 0, 1$.*
- b) *For any torus T , $\underline{R}^q \pi^X_* \underline{\mathbb{T}}$ is 1-motivic for $q = 0, 1$.*

Proof. We apply Lemma 11.2.3 a) in the following situation: $\mathcal{A} = \text{HI}_{\text{ét}}$, $\mathcal{B} = \text{Shv}_0$, $H^i(X) = R^i \pi^X_* L$ in case a), $\mathcal{B} = \text{Shv}_1$, $H^i(X) = R^i \pi^X_* \underline{\mathbb{T}}$ in case b). The smooth case is trivial in a) and the lattice assertions follow from lemmas 13.2.1 and 13.2.2 a). In b), the smooth case follows from Proposition 3.4.1. □

13.6. $L_1\text{Alb}(X)$ and the Albanese schemes. We now compute the 1-motive $L_1\text{Alb}(X) = [L_1 \rightarrow G_1]$ in important special cases. This is done in the following three propositions; in particular, we shall show that it always “is” a Deligne 1-motive. Note that, by definition of a 1-motive with cotorsion, the pair (L_1, G_1) is determined only up to a q.i.: the last sentence means that we may choose this pair such that G_1 is connected (and then it is uniquely determined).

13.6.1. Proposition. *Let $X \in \text{Sch}(k)$. Then*

a) $\mathcal{H}_i(\text{LAlb}(X)) = 0$ for $i < 0$.

b) Let $\mathcal{F}_X = \mathcal{H}_0(\text{Tot LAlb}(X))$. Then \mathcal{F}_X corepresents the functor

$$\begin{aligned} \text{Shv}_1 &\rightarrow \text{Ab} \\ \mathcal{F} &\mapsto \mathcal{F}_{\text{éh}}(X) \text{ (see Def. 11.2.1)} \end{aligned}$$

via the composition

$$\alpha^*M(X) \rightarrow \text{Tot LAlb}(X) \rightarrow \mathcal{F}_X[0].$$

Moreover, we have an exact sequence, for any representative $[L_1 \xrightarrow{u_1} G_1]$ of $L_1\text{Alb}(X)$:

$$(13.2) \quad L_1 \xrightarrow{u_1} G_1 \rightarrow \mathcal{F}_X \rightarrow \mathbb{Z}\pi_0(X) \rightarrow 0.$$

c) Let $\mathcal{A}_{X/k}^{\text{éh}} := \Omega(\mathcal{F}_X)$ (cf. Proposition 3.6.5). Then $\mathcal{A}_{X/k}^{\text{éh}}$ corepresents the functor

$$\begin{aligned} {}^t\text{AbS} &\rightarrow \text{Ab} \\ G &\mapsto \underline{G}_{\text{éh}}(X). \end{aligned}$$

Moreover we have an epimorphism

$$(13.3) \quad \mathcal{A}_{X/k}^{\text{éh}} \twoheadrightarrow \mathcal{A}_{X_{\text{red}}/k}.$$

d) If X_{red} is strictly reduced (Def. 13.3.1) or normal, (13.3) is an isomorphism.

Proof. a) is proven as in Proposition 11.3.2 by blow-up induction (reduction to the smooth case). If $\mathcal{F} \in \text{Shv}_1$, we have

$$\text{Hom}_{\text{DM}_{-,\text{ét}}^{\text{eff}}}(\alpha^*M(X), \mathcal{F}) = \mathcal{F}_{\text{éh}}(X)$$

by Propositions 3.8.1, 11.2.2 and D.1.4. The latter group coincides with $\text{Hom}_{\text{Shv}_1}(\mathcal{F}_X, \mathcal{F})$ by (5.3) and a), hence b); the exact sequence follows from Proposition 3.10.2. The sheaf $\mathcal{A}_{X/k}^{\text{éh}}$ clearly corepresents the said functor; the map then comes from the obvious natural transformation in G : $G(X_{\text{red}}) \rightarrow G_{\text{éh}}(X)$ and its surjectivity follows from Lemma 13.3.4 a), hence c). d) follows from Lemma 13.3.4 b), Theorem 13.4.1 and the universal property of $\mathcal{A}_{X/k}$. \square

13.6.2. Remark. One could christen \mathcal{F}_X and $\mathcal{A}_{X/k}^{\text{éh}}$ the *universal 1-motivic sheaf* and the *éh-Albanese scheme* of X .

13.6.3. Proposition. a) *The sheaves \mathcal{F}_X and $\mathcal{A}_{X/k}^{\text{éh}}$ have π_0 equal to $\mathbb{Z}[\pi_0(X)]$; in particular, $\mathcal{A}_{X/k}^{\text{éh}} \in \text{AbS}$.*

b) *In (13.2), the composition $L_1 \xrightarrow{u_1} G_1 \rightarrow \pi_0(G_1)$ is surjective.*

c) *One may choose $L_1 \text{Alb}(X) \simeq [L_1 \rightarrow G_1]$ with G_1 connected (in other words, $L_1 \text{Alb}(X)$ is a Deligne 1-motive).*

Proof. In a), it suffices to prove the first assertion for \mathcal{F}_X : then it follows from its universal property and Lemma 13.2.1 a). The second assertion of a) is obvious.

b) Let $0 \rightarrow L'_1 \rightarrow G'_1 \rightarrow \mathcal{F}_X \rightarrow \mathbb{Z}[\pi_0(X)] \rightarrow 0$ be the normalised presentation of \mathcal{F}_X given by Proposition 3.2.3. We have a commutative diagram

$$\begin{array}{ccccccccc}
 0 & \longrightarrow & L'_1 & \longrightarrow & G'_1 & \longrightarrow & \mathcal{F}_X & \longrightarrow & \mathbb{Z}[\pi_0(X)] & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & \parallel & & \parallel & & \\
 0 & \longrightarrow & \overline{u_1(L_1)} & \xrightarrow{\overline{u_1}} & \overline{G_1} & \longrightarrow & \mathcal{F}_X & \longrightarrow & \mathbb{Z}[\pi_0(X)] & \longrightarrow & 0 \\
 & & \uparrow & & \uparrow & & \parallel & & \parallel & & \\
 & & L_1 & \xrightarrow{u_1} & G_1 & \longrightarrow & \mathcal{F}_X & \longrightarrow & \mathbb{Z}[\pi_0(X)] & \longrightarrow & 0
 \end{array}$$

with $\overline{u_1(L_1)} = u_1(L_1)/F$ and $\overline{G_1} = G_1/F$, where F is the torsion subgroup of $u_1(L_1)$. Indeed, $\text{Ext}(G'_1, \overline{u_1(L_1)}) = 0$ so we get the downwards vertical maps as in the proof of Proposition 3.2.3. By uniqueness of the normalised presentation, G'_1 maps onto $\overline{G_1}^0$. A diagram chase then shows that the composition

$$\overline{u_1(L_1)} \xrightarrow{\overline{u_1}} \overline{G_1} \rightarrow \pi_0(\overline{G_1})$$

is onto, and another diagram chase shows the same for u_1 .

c) The pull-back diagram

$$\begin{array}{ccc}
 L_1^0 & \longrightarrow & G_1^0 \\
 \downarrow & & \downarrow \\
 L_1 & \longrightarrow & G_1
 \end{array}$$

is a quasi-isomorphism in ${}_t\mathcal{M}_1^{\text{eff}}$, thanks to b). \square

13.6.4. Lemma. *Suppose that k is algebraically closed. Let $[L_1 \xrightarrow{u_1} G_1]$ be the Deligne 1-motive that lies in the q.i. class of $L_1 \text{Alb}(X)$, thanks*

to Proposition 13.6.3 c), and let L be a lattice.

a) We have an isomorphism

$$\mathrm{Hom}_{D^b(\mathcal{M}_1)}(\mathrm{LAlb}(X), L[1]) \xrightarrow{\sim} \mathrm{Hom}(L_1, L).$$

b) The map

$$\mathrm{Hom}_{D^b(\mathcal{M}_1)}(\mathrm{LAlb}(X), L[j]) \rightarrow \mathrm{Hom}_{\mathrm{DM}_{\mathrm{gm}, \mathrm{et}}^{\mathrm{eff}}}(M_{\mathrm{et}}(X), L[j]) = H_{\mathrm{eh}}^j(X, L)$$

induced by a_X (8.3) is an isomorphism for $j = 0, 1$ (see Prop. 11.2.2 for the last equality).

Proof. a) From the spectral sequence (11.1), we get an exact sequence

$$\begin{aligned} 0 \rightarrow \mathrm{Ext}^1(\mathrm{L}_0\mathrm{Alb}(X), L) &\rightarrow \mathrm{Hom}_{D^b(\mathcal{M}_1)}(\mathrm{LAlb}(X), L[1]) \\ &\rightarrow \mathrm{Hom}(\mathrm{L}_1\mathrm{Alb}(X), L) \rightarrow \mathrm{Ext}^2(\mathrm{L}_0\mathrm{Alb}(X), L). \end{aligned}$$

Since the two Ext are 0, the middle map is an isomorphism. Since $[L_1 \rightarrow G_1]$ is a Deligne 1-motive, $\mathrm{Hom}(\mathrm{L}_1\mathrm{Alb}(X), L)$ is isomorphic to $\mathrm{Hom}(L_1, L)$.

b) By blow-up induction (Lemma 11.2.3) we reduce to X smooth. If $j = 0$, the result is trivial; if $j = 1$, it is also trivial because both sides are 0 (by a) and Corollary 10.2.3 for the left hand side). \square

13.6.5. Proposition. *Keep the above notation $\mathrm{L}_1\mathrm{Alb}(X) = [L_1 \xrightarrow{u_1} G_1]$.*

a) We have an isomorphism

$$L_1 \simeq \mathcal{H}om(\underline{R}^1\pi_*\mathbb{Z}, \mathbb{Z})$$

(cf. Proposition 13.5.1).

b) We have a canonical isomorphism

$$G_1/(L_1)_{\mathrm{Zar}} \xrightarrow{\sim} (\mathcal{A}_{X/k}^{\mathrm{eh}})^0$$

where $(L_1)_{\mathrm{Zar}}$ is the Zariski closure of the image of L_1 in G_1 and $\mathcal{A}_{X/k}^{\mathrm{eh}}$ was defined in Proposition 13.6.1 ($(\mathcal{A}_{X/k}^{\mathrm{eh}})^0$ corepresents the functor $\mathrm{SA}b \ni G \mapsto G_{\mathrm{eh}}(X)$).

Proof. For the computations, we may assume k algebraically closed.

Let L be a lattice. By Lemma 13.6.4, we have an isomorphism

$$H_{\mathrm{eh}}^1(X, L) \simeq \mathrm{Hom}(L_1, L).$$

This gives a), since we obviously have $H_{\mathrm{eh}}^1(X, L) = H_{\mathrm{et}}^0(k, \underline{R}^1\pi_*L) = \underline{R}^1\pi_*\mathbb{Z} \otimes L$ by Proposition 13.5.1 a).

b) follows directly from the definition of $\mathcal{A}_{X/k}^{\mathrm{eh}}$. \square

13.6.6. Corollary. *Let $L_1\mathrm{Alb}(X) = [L_1 \rightarrow G_1]$, as a Deligne 1-motive.*

a) If X is proper, then G_1 is an abelian variety.

b) If X is normal, then $L_1\mathrm{Alb}(X) = [0 \rightarrow \mathcal{A}_{X/k}^0]$.

c) If X is normal and proper then $R^1\mathrm{Pic}(X) = [0 \rightarrow \mathrm{Pic}_{X/k}^0]$ is an abelian variety with dual the Serre Albanese $L_1\mathrm{Alb}(X) = [0 \rightarrow \mathcal{A}_{X/k}^0]$.

Proof. a) is seen easily by blow-up induction, by reducing to the smooth projective case (Corollary 10.2.3). b) follows from Proposition 13.6.5 a), b), Lemma 13.2.2 c) and Proposition 13.6.1 d). c) follows immediately from a) and b). \square

13.7. $L_1\mathrm{Alb}(X)$ and $\mathrm{Alb}^-(X)$ for X normal. In 13.7.2, we prove that these two 1-motives are isomorphic. We begin with a slight improvement of Theorem 13.4.1 in the case of semi-abelian schemes:

13.7.1. Lemma. *Let X be normal, and let X_\bullet be a smooth hyperenvelope (cf. Lemma 13.4.2). Then we have*

$$\mathcal{A}_{X/k} = \mathrm{Coker}(\mathcal{A}_{X_1} \xrightarrow{(p_0)^* - (p_1)^*} \mathcal{A}_{X_0})$$

and

$$(\mathcal{A}_{X/k})^0 = \mathrm{Coker}(\mathcal{A}_{X_1}^0 \xrightarrow{(p_0)^* - (p_1)^*} \mathcal{A}_{X_0}^0).$$

Proof. The first isomorphism follows from Lemma 13.4.2 applied with Y running through torsors under semi-abelian varieties. To deduce the second isomorphism, consider the short exact sequence of complexes

$$0 \rightarrow \mathcal{A}_{X_\bullet/k}^0 \rightarrow \mathcal{A}_{X_\bullet/k} \rightarrow \mathbb{Z}[\pi_0(X_\bullet)] \rightarrow 0$$

and the resulting long exact sequence

$$(13.4) \quad H_1(\mathbb{Z}[\pi_0(X_\bullet)]) \rightarrow H_0(\mathcal{A}_{X_\bullet/k}^0) \rightarrow H_0(\mathcal{A}_{X_\bullet/k}) \rightarrow H_0(\mathbb{Z}[\pi_0(X_\bullet)]) \rightarrow 0.$$

For any $i \geq 0$, $\mathbb{Z}[\pi_0(X_i)]$ is \mathbb{Z} -dual to the Galois module $E_1^{i,0}$, where $E_1^{p,q} = H_{\mathrm{\acute{e}t}}^q(X_p \times_k \bar{k}, \mathbb{Z})$ is the E_1 -term associated to the simplicial spectral sequence for $X_\bullet \times_k \bar{k}$. Since $H_{\mathrm{\acute{e}t}}^1(X_p \times_k \bar{k}, \mathbb{Z}) = 0$ for all $p \geq 0$, we get

$$H_i(\mathbb{Z}[\pi_0(X_\bullet)]) \simeq (H_{\mathrm{\acute{e}t}}^i(X_\bullet \times_k \bar{k}, \mathbb{Z}))^\vee \text{ for } i = 0, 1.$$

By Proposition 11.2.2, these étale cohomology groups may be replaced by éh cohomology groups. By Proposition 11.2.5, we then have

$$H_{\mathrm{\acute{e}t}}^i(X_\bullet \times_k \bar{k}, \mathbb{Z}) \simeq H_{\mathrm{\acute{e}h}}^i(X \times_k \bar{k}, \mathbb{Z}).$$

Now, by Lemma 13.2.1 a), $H_{\text{eh}}^0(X \times_k \bar{k}, \mathbb{Z})$ is \mathbb{Z} -dual to $\mathbb{Z}[\pi_0(X)]$, and by Lemma 13.2.2 c), $H_{\text{eh}}^1(X \times_k \bar{k}, \mathbb{Z}) = 0$ because X is normal. Hence (13.4) yields a short exact sequence

$$0 \rightarrow H_0(\mathcal{A}_{X_\bullet/k}^0) \rightarrow \mathcal{A}_{X/k} \rightarrow \mathbb{Z}[\pi_0(X)] \rightarrow 0$$

which identifies $H_0(\mathcal{A}_{X_\bullet/k}^0)$ with $\mathcal{A}_{X/k}^0$. \square

13.7.2. Proposition. *If X is normal, $R^1\text{Pic}(X)$ and $L_1\text{Alb}(X)$ are isomorphic, respectively, to the 1-motives $\text{Pic}^+(X)$ and $\text{Alb}^-(X)$ defined in [8, Ch. 4-5].*

Proof. Let \bar{X} be a normal compactification of X ; choose a smooth hyperenvelope X_\bullet of X along with \bar{X}_\bullet , a smooth compactification with normal crossing boundary Y_\bullet such that $\bar{X}_\bullet \rightarrow \bar{X}$ is an hyperenvelope. Now we have, in the notation of [8, 4.2], a commutative diagram with exact rows

$$\begin{array}{ccccccc} 0 & \longrightarrow & \text{Pic}_{X/k}^0 & \longrightarrow & \text{Pic}_{X_0/k}^0 & \longrightarrow & \text{Pic}_{X_1/k}^0 \\ & & \uparrow & & \uparrow & & \uparrow \\ 0 & \longrightarrow & \text{Div}_{Y_\bullet}^0(\bar{X}_\bullet) & \longrightarrow & \text{Div}_{Y_0}^0(\bar{X}_0) & \longrightarrow & \text{Div}_{Y_1}^0(\bar{X}_1) \end{array}$$

where $\text{Pic}^+(X) = [\text{Div}_{Y_\bullet}^0(\bar{X}_\bullet) \rightarrow \text{Pic}_{X/k}^0]$ since \bar{X} is normal. Taking Cartier duals we get an exact sequence of 1-motives

$$[0 \rightarrow \mathcal{A}_{X_1/k}^0] \xrightarrow{(p_0)^* - (p_1)^*} [0 \rightarrow \mathcal{A}_{X_0/k}^0] \rightarrow \text{Alb}^-(X) \rightarrow 0.$$

Thus $\text{Alb}^-(X) = [0 \rightarrow \mathcal{A}_{X/k}^0]$ by Lemma 13.7.1. We conclude by Corollary 13.6.6 b) since X is normal and $L_1\text{Alb}(X) = [0 \rightarrow \mathcal{A}_{X/k}^0]$. \square

13.7.3. Remarks. 1) Note that, while $L_0\text{Alb}(X)$ and $L_1\text{Alb}(X)$ are Deligne 1-motives, the same is not true of $L_2\text{Alb}(X)$ in general, already for X smooth projective (see Corollary 10.2.3).

2) One could make use of Proposition 11.3.1 to compute $L_i\text{Alb}(X)$ for singular X and $i > 1$. However, $H_{\text{eh}}^i(X, \mathbb{G}_m)_{\mathbb{Q}}$ can be non-zero also for $i \geq 2$, therefore a precise computation for X singular and higher dimensional appears to be difficult. We did completely the case of curves in Sect. 12.

13.8. $\text{RPic}(X)$ and $H_{\text{eh}}^*(X, \mathbb{G}_m)$. By definition of RPic , we have a morphism in $\text{DM}_{-, \text{ét}}^{\text{eff}}$

$$\begin{aligned} \text{Tot RPic}(X) &= \alpha^* \text{Hom}_{\text{Nis}}(M(X), \mathbb{Z}(1)) \\ &\rightarrow \text{Hom}_{\text{ét}}(M_{\text{ét}}(X), \mathbb{Z}(1)) = \underline{R}\pi_*^X \mathbb{G}_m[-1]. \end{aligned}$$

This gives homomorphisms

$$(13.5) \quad \mathcal{H}^i(\mathrm{Tot} \mathrm{RPic}(X)) \rightarrow \underline{R}^{i-1}\pi_*^X \mathbb{G}_m, \quad i \geq 0.$$

13.8.1. Proposition. *For $i \leq 2$, (13.5) is an isomorphism.*

Proof. By blow-up induction, we reduce to the smooth case, where it follows from Hilbert's theorem 90. \square

13.9. $H_{\mathrm{\acute{e}t}}^1(X, \mathbb{G}_m)$ and $H_{\mathrm{eh}}^1(X, \mathbb{G}_m)$. In this subsection, we assume $\pi^X : X \rightarrow \mathrm{Spec} k$ *proper*. Recall that, then, the étale sheaf associated to the presheaf

$$U \mapsto \mathrm{Pic}(X \times U)$$

is representable by a k -group scheme $\mathrm{Pic}_{X/k}$ locally of finite type (Grothendieck-Murre [66]). Its connected component $\mathrm{Pic}_{X/k}^0$ is an extension of a semi-abelian variety by a unipotent subgroup \mathcal{U} . By homotopy invariance of $\underline{R}^1\pi_*^X \mathbb{G}_m$, we get a map

$$(13.6) \quad \mathrm{Pic}_{X/k} / \mathcal{U} \rightarrow \underline{R}^1\pi_*^X \mathbb{G}_m.$$

Recall that the right hand side is a 1-motivic sheaf by Proposition 13.5.1. We have:

13.9.1. Proposition. *This map is injective with lattice cokernel.*

Proof. Consider multiplication by an integer $n > 1$ on both sides. Using the Kummer exact sequence, Proposition 13.1.1 and Lemma 13.3.4 c), we find that (13.6) is an isomorphism on n -torsion and injective on n -cotorsion. The conclusion then follows from Proposition 3.6.6. \square

13.10. $\mathrm{R}^1\mathrm{Pic}(X)$ and $\mathrm{Pic}^+(X)$ for X proper.

13.10.1. Theorem. *For X proper, the composition*

$$\mathrm{Pic}_{X/k} / \mathcal{U} \rightarrow \underline{R}^1\pi_*^X \mathbb{G}_m \rightarrow \mathcal{H}^2(\mathrm{Tot} \mathrm{RPic}(X))$$

where the first map is (13.6) and the second one is the inverse of the isomorphism (13.5), induces an isomorphism

$$\mathrm{Pic}^+(X) \xrightarrow{\sim} \mathrm{R}^1\mathrm{Pic}(X)$$

where $\mathrm{Pic}^+(X)$ is the 1-motive defined in [8, Ch. 4].

Proof. Proposition 3.10.2 yields an exact sequence

$$L^1 \rightarrow G^1 \rightarrow \mathcal{H}^2(\mathrm{Tot} \mathrm{RPic}(X)) \rightarrow L^2$$

where we write $\mathrm{R}^i\mathrm{Pic}(X) = [L^i \rightarrow G^i]$. Propositions 13.8.1 and 13.9.1 then imply that the map of Theorem 13.10.1 induces an isomorphism $\mathrm{Pic}_{X/k}^0 / \mathcal{U} \xrightarrow{\sim} G^1$. The conclusion follows, since on the one hand $\mathrm{Pic}^+(X) \simeq [0 \rightarrow \mathrm{Pic}_{X/k}^0 / \mathcal{U}]$ by [8, Lemma 5.1.2 and Remark 5.1.3], and on the other hand the dual of Corollary 13.6.6 a) says that $L^1 = 0$. \square

13.10.2. **Corollary.** *For X proper there is a canonical isomorphism*

$$L_1 \mathrm{Alb}(X) \xrightarrow{\sim} \mathrm{Alb}^-(X). \quad \square$$

13.11. **The Borel-Moore variant.** Let $X \in \mathrm{Sch}$ be provided with a compactification \overline{X} and closed complement $Z \xrightarrow{i} \overline{X}$. The relative Picard functor is then representable by a k -group scheme locally of finite type $\mathrm{Pic}_{\overline{X},Z}$, and we shall informally denote by \mathcal{U} its unipotent radical. Similarly to (13.5) and (13.6), we have two canonical maps

$$(13.7) \quad \mathcal{H}^2(\mathrm{Tot} \mathrm{RPic}^c(X)) \rightarrow \mathrm{Pic}_{\overline{X},Z}^{\mathrm{eh}} \leftarrow \mathrm{Pic}_{\overline{X},Z} / \mathcal{U}$$

where $\mathrm{Pic}_{\overline{X},Z}^{\mathrm{eh}}$ is by definition the 1-motivic sheaf associated to the presheaf $U \mapsto H_{\mathrm{eh}}^1(\overline{X} \times U, (\mathbb{G}_m)_{\overline{X} \times U} \rightarrow i_*(\mathbb{G}_m)_{Z \times U})$ (compare [8, 2.1]). Indeed, the latter group is canonically isomorphic to

$$\mathrm{Hom}_{\mathrm{DM}_{-,\mathrm{et}}^{\mathrm{eff}}} (M^c(X \times U), \mathbb{Z}(1)[2])$$

via the localisation exact triangle. From Theorem 13.10.1 and Proposition 11.4.2 b), we then deduce:

13.11.1. **Theorem.** *The maps (13.7) induce an isomorphism*

$$R^1 \mathrm{Pic}^c(X) \simeq [0 \rightarrow \mathrm{Pic}^0(\overline{X}, Z) / \mathcal{U}]. \quad \square$$

The following is a sequel of Proposition 11.5.2:

13.11.2. **Corollary.** *Let $X \in \mathrm{Sch}(k)$ of dimension d . Then:*

- a) $L_1 \mathrm{Alb}^c(X) = [L_1 \rightarrow A_1]$, where A_1 is an abelian variety. In particular, $L_1 \mathrm{Alb}^c(X)$ is a Deligne 1-motive.
- b) If X is normal connected and not proper, let \overline{X} be a normal compactification of X . Then $\mathrm{rank} L_1 = \#\pi_0(\overline{X} - X) - 1$.

Proof. a) follows immediately from Theorem 13.11.1. For b), consider the complex of discrete parts associated to the exact sequence (9.1): we get with obvious notation an almost exact sequence

$$L_1(\overline{X}) \rightarrow L_1(X) \rightarrow L_0(\overline{X} - X) \rightarrow L_0(\overline{X}) \rightarrow L_0(X)$$

where “almost exact” means that its homology is finite. The last group is 0 and $L_0(\overline{X}) = \mathbb{Z}$; on the other hand, $L_1(\overline{X}) = 0$ by Corollary 13.6.6 b). Hence the claim. \square

13.11.3. **Remarks** (on Corollary 11.3.2).

- 1) In fact, $A_1 = 0$ in a) if X is smooth and quasi-affine of dimension > 1 : see Corollary 14.6.2. This contrasts sharply with Theorem 12.2.3 for smooth curves.
- 2) As a consequence of the statement of b) we see that in b), the number of connected components of $\overline{X} - X$ only depends on X . Here is an

elementary proof of this fact: let \overline{X}' be another normal compactification and \overline{X}'' the closure of X in $\overline{X} \times \overline{X}'$. Then the two maps $\overline{X}'' \rightarrow \overline{X}$ and $\overline{X}'' \rightarrow \overline{X}'$ have connected fibres by Zariski's main theorem, thus $\overline{X} - X$ and $\overline{X}' - X$ have the same number of connected components as $\overline{X}'' - X$. (The second author is indebted to Marc Hindry for a discussion leading to this proof.)

We shall also need the following computation in the next subsection.

13.11.4. Theorem. *Let \overline{X} be smooth and proper, $Z \subset \overline{X}$ a divisor with normal crossings and $X = \overline{X} - Z$. Let Z_1, \dots, Z_r be the irreducible components of Z and set*

$$Z^{(p)} = \begin{cases} \overline{X} & \text{if } p = 0 \\ \coprod_{i_1 < \dots < i_p} Z_{i_1} \cap \dots \cap Z_{i_p} & \text{if } p > 0. \end{cases}$$

Let $\mathrm{NS}_c^{(p)}(X)$ (resp. $\mathrm{Pic}_c^{(p)}(X)$, $T_c^{(p)}(X)$) be the cohomology (resp. the connected component of the cohomology) in degree p of the complex

$$\dots \rightarrow \mathrm{NS}(Z^{(p-1)}) \rightarrow \mathrm{NS}(Z^{(p)}) \rightarrow \mathrm{NS}(Z^{(p+1)}) \rightarrow \dots$$

(resp.

$$\dots \rightarrow \mathrm{Pic}^0(Z^{(p-1)}) \rightarrow \mathrm{Pic}^0(Z^{(p)}) \rightarrow \mathrm{Pic}^0(Z^{(p+1)}) \rightarrow \dots$$

$$\dots \rightarrow \mathbb{R}_{\pi_0(Z^{(p-1)})/k} \mathbb{G}_m \rightarrow \mathbb{R}_{\pi_0(Z^{(p)})/k} \mathbb{G}_m \rightarrow \mathbb{R}_{\pi_0(Z^{(p+1)})/k} \mathbb{G}_m \rightarrow \dots).$$

Then, for all $n \geq 0$, $R^n \mathrm{Pic}^c(X)$ is of the form $[\mathrm{NS}_c^{(n-2)}(X) \xrightarrow{u^n} G_c^{(n)}]$, where $G_c^{(n)}$ is an extension of $\mathrm{Pic}_c^{(n-1)}(X)$ by $T_c^{(n)}(X)$.

Proof. A standard argument (compare e.g. [33, 3.3]) yields a spectral sequence of cohomological type in ${}^t\mathcal{M}_1$:

$$E_1^{p,q} = R^q \mathrm{Pic}^c(Z^{(p)}) \Rightarrow R^{p+q} \mathrm{Pic}^c(X).$$

By Corollary 10.6.1, we have $E_2^{p,2} = [\mathrm{NS}_c^{(p)}(X) \rightarrow 0]$, $E_2^{p,1} = [0 \rightarrow \mathrm{Pic}_c^{(p)}(X)]$ and $E_2^{p,0} = [0 \rightarrow T_c^{(p)}(X)]$. By Proposition C.9.1, all d_2 differentials are 0, hence the theorem. \square

13.11.5. Corollary. *With notation as in Theorem 13.11.4, the complex $R\mathrm{Pic}(M^c(X)(1)[2])$ is q.i. to*

$$\dots \rightarrow [\mathbb{Z}^{\pi_0(Z^{(p-2)})} \rightarrow 0] \rightarrow \dots$$

In particular, $R^0 \mathrm{Pic}(M^c(X)(1)[2]) = R^1 \mathrm{Pic}(M^c(X)(1)[2]) = 0$ and $R^2 \mathrm{Pic}(M^c(X)(1)[2]) = [\mathbb{Z}^{\pi_0^c(X)} \rightarrow 0]$ (see Definition 11.5.1).

Proof. This follows from Theorem 13.11.4 via the formula $M^c(X \times \mathbb{P}^1) = M^c(X) \oplus M^c(X)(1)[2]$, noting that $\overline{X} \times \mathbb{P}^1$ is a smooth compactification of $X \times \mathbb{P}^1$ with $\overline{X} \times \mathbb{P}^1 - X \times \mathbb{P}^1$ a divisor with normal crossings with components $Z_i \times \mathbb{P}^1$, and

$$\begin{aligned} \mathrm{NS}(Z^{(p)} \times \mathbb{P}^1) &= \mathrm{NS}(Z^{(p)}) \oplus \mathbb{Z}^{\pi_0(Z^{(p)})} \\ \mathrm{Pic}^0(Z^{(p)} \times \mathbb{P}^1) &= \mathrm{Pic}^0(Z^{(p)}) \\ \pi_0(Z^{(p)} \times \mathbb{P}^1) &= \pi_0(Z^{(p)}). \end{aligned}$$

□

13.11.6. Remark. Let X be arbitrary, and filter it by its successive singular loci, *i.e.*

$$X = X^{(0)} \supset X^{(1)} \supset \dots$$

where $X^{(i+1)} = X_{\mathrm{sing}}^{(i)}$. Then we have a spectral sequence of cohomological type in ${}^t\mathcal{M}_1$

$$E_2^{p,q} = \mathrm{R}^{p+q}\mathrm{Pic}^c(X^{(q)} - X^{(q+1)}) \Rightarrow \mathrm{R}^{p+q}\mathrm{Pic}^c(X)$$

in which the E_2 -terms involve smooth varieties. This qualitatively reduces the computation of $\mathrm{R}^*\mathrm{Pic}^c(X)$ to the case of smooth varieties, but the actual computation may be complicated; we leave this to the interested reader.

13.12. $\mathrm{L}_1\mathrm{Alb}^*$ and Alb^+ .

13.12.1. Lemma. *Let $n > 0$ and $Z \in \mathrm{Sch}$ of dimension $< n$; then*

$$\begin{aligned} \mathrm{R}^i\mathrm{Pic}(M(Z)^*(n)[2n]) &= 0 \text{ for } i \leq 1 \\ \mathrm{R}^2\mathrm{Pic}(M(Z)^*(n)[2n]) &= [\mathbb{Z}^{\pi_0^c(Z^{[n-1]})} \rightarrow 0] \end{aligned}$$

where $Z^{[n-1]}$ is the disjoint union of the irreducible components of Z of dimension $n-1$.

Proof. Let

$$\begin{array}{ccc} \tilde{T} & \longrightarrow & \tilde{Z} \\ \downarrow & & \downarrow \\ T & \longrightarrow & Z \end{array}$$

be an abstract blow-up square, with \tilde{Z} smooth and $\dim T, \dim \tilde{T} < \dim Z$. By Lemma 9.2.1, $M(T)^*(n-2)[2n-4]$ and $M(\tilde{T})^*(n-2)[2n-4]$ are effective, so by Proposition 8.1.2, the exact triangle

$$\begin{aligned} \mathrm{RPic}(M(\tilde{T})^*(n)[2n]) &\rightarrow \mathrm{RPic}(M(\tilde{Z})^*(n)[2n]) \oplus \mathrm{RPic}(M(T)^*(n)[2n]) \\ &\rightarrow \mathrm{RPic}(M(Z)^*(n)[2n]) \xrightarrow{+1} \end{aligned}$$

degenerates into an isomorphism

$$\mathrm{RPic}(M(\tilde{Z})^*(n)[2n]) \xrightarrow{\sim} \mathrm{RPic}(M(Z)^*(n)[2n]).$$

The lemma now follows from Corollary 13.11.5 by taking for \tilde{Z} a desingularisation of $Z^{[n-1]}$ and for T the union of the singular locus of Z and its irreducible components of dimension $< n - 1$ (note that $M(\tilde{Z})^*(n)[2n] \simeq M^c(\tilde{Z})(1)[2]$). \square

13.12.2. Lemma. *Let \overline{X} a proper smooth scheme with a pair Y and Z of disjoint closed (reduced) subschemes of pure codimension 1 in \overline{X} . We then have*

$$\mathrm{R}^1\mathrm{Pic}(\overline{X} - Z, Y) \cong \mathrm{Pic}^+(\overline{X} - Z, Y)$$

(see [8, 2.2.1] for the definition of relative Pic^+).

Proof. The following exact sequence provides the weight filtration

$$0 \rightarrow \mathrm{R}^1\mathrm{Pic}(\overline{X}, Y) \rightarrow \mathrm{R}^1\mathrm{Pic}(\overline{X} - Z, Y) \rightarrow \mathrm{R}^2\mathrm{Pic}_Z(\overline{X}, Y)$$

where $\mathrm{R}^1\mathrm{Pic}(\overline{X}, Y) \cong \mathrm{R}^1\mathrm{Pic}^c(\overline{X} - Y) \cong [0 \rightarrow \mathrm{Pic}^0(\overline{X}, Y)]$ by Theorem 13.11.1 (here $\mathcal{U} = 0$ since \overline{X} is smooth). Also $\mathrm{R}^2\mathrm{Pic}_Z(\overline{X}, Y) \cong \mathrm{R}^2\mathrm{Pic}_Z(\overline{X}) = [\mathrm{Div}_Z(\overline{X}) \rightarrow 0]$ from 8.2.2: thus the discrete part of $\mathrm{R}^1\mathrm{Pic}(\overline{X} - Z, Y)$ is given by a subgroup D of $\mathrm{Div}_Z(\overline{X}) = \mathrm{Div}_Z(\overline{X}, Y)$.

It remains to identify the map $u : D \rightarrow \mathrm{Pic}^0(\overline{X}, Y)$. Using now the exact sequence

$$\mathrm{R}^0\mathrm{Pic}(Y) \rightarrow \mathrm{R}^1\mathrm{Pic}(\overline{X} - Z, Y) \rightarrow \mathrm{R}^1\mathrm{Pic}(\overline{X} - Z) \rightarrow \mathrm{R}^1\mathrm{Pic}(Y)$$

where $\mathrm{R}^i\mathrm{Pic}(Y)$ is of weight < 0 for $i \leq 1$ (11.3.2 and 13.6.6), we get that u is the canonical lifting of the map of the 1-motive $\mathrm{R}^1\mathrm{Pic}(\overline{X} - Z)$ described in 10.6.1. Thus $D = \mathrm{Div}_Z^0(\overline{X}, Y)$ and the claimed isomorphism is clear. \square

This proof also gives:

13.12.3. Corollary. *We have*

$$[\mathrm{Div}_Z^0(\overline{X}, Y) \rightarrow 0] = \mathrm{Ker}(\mathrm{R}^2\mathrm{Pic}_Z(\overline{X}, Y) \rightarrow \mathrm{R}^2\mathrm{Pic}(\overline{X}, Y)).$$

We shall need:

13.12.4. Theorem (Relative duality). *Let \overline{X} , Y and Z be as above and further assume that \overline{X} is n -dimensional. Then*

$$M(\overline{X} - Z, Y)^*(n)[2n] \cong M(\overline{X} - Y, Z)$$

and therefore

$$\mathrm{RPic}^*(\overline{X} - Z, Y) \cong \mathrm{RPic}(\overline{X} - Y, Z)$$

and dually for LAlb .

Proof. See [6]. □

13.12.5. Corollary. *Let Z be a divisor in \overline{X} such that $Z \cap Y = \emptyset$. There exists a “cycle class” map η fitting in the following commutative diagram*

$$\begin{array}{ccc} \mathrm{R}^2\mathrm{Pic}(M(Z)^*(n)[2n]) & \xrightarrow{\eta} & \mathrm{R}^2\mathrm{Pic}^c(\overline{X} - Y) \\ \parallel \downarrow & & \parallel \downarrow \\ \mathrm{R}^2\mathrm{Pic}_Z(\overline{X}, Y) & \longrightarrow & \mathrm{R}^2\mathrm{Pic}(\overline{X}, Y) \end{array}$$

Writing $Z = \cup Z_i$ as union of its irreducible components we have that η on Z_i is the “fundamental class” of Z_i in \overline{X} modulo algebraic equivalence.

Proof. We have a map $M(Z) \rightarrow M(\overline{X} - Y)$, and the vertical isomorphisms in the following commutative square

$$\begin{array}{ccc} M(\overline{X} - Y)^*(n)[2n] & \longrightarrow & M(Z)^*(n)[2n] \\ \parallel \uparrow & & \parallel \uparrow \\ M(\overline{X}, Y) & \longrightarrow & M^Z(\overline{X}, Y) \end{array}$$

are given by relative duality. □

13.12.6. Theorem. *For $X \in \mathrm{Sch}$ we have*

$$\mathrm{R}^1\mathrm{Pic}^*(X) \cong \mathrm{Pic}^-(X).$$

Proof. We are left to consider $X \in \mathrm{Sch}$ purely of dimension n with the following associated set of data and conditions.

For the irreducible components X_1, \dots, X_r of X we let \tilde{X} be a desingularisation of $\coprod X_i$, $S := X_{\mathrm{sing}} \cup \bigcup_{i \neq j} S_i \cap S_j$ and \tilde{S} the inverse image of S in \tilde{X} . We let \overline{X} be a smooth proper compactification with normal crossing divisor Y . Let \overline{S} denote the Zariski closure of \tilde{S} in \overline{X} . Assume that $Y + \overline{S}$ is a reduced normal crossing divisor in \overline{X} . Finally denote by Z the union of all compact components of divisors in \tilde{S} (cf. [8, 2.2]).

We have an exact sequence coming from the abstract blow-up square associated to the above picture:

$$\begin{aligned} \cdots \rightarrow \mathrm{R}^1\mathrm{Pic}^*(\tilde{X}) \oplus \mathrm{R}^1\mathrm{Pic}(M(S)^*(n)[2n]) &\rightarrow \mathrm{R}^1\mathrm{Pic}^*(X) \\ \rightarrow \mathrm{R}^2\mathrm{Pic}(M(\tilde{S})^*(n)[2n]) &\rightarrow \mathrm{R}^2\mathrm{Pic}^*(\tilde{X}) \oplus \mathrm{R}^2\mathrm{Pic}(M(S)^*(n)[2n]) \end{aligned}$$

Now:

- the first map is injective (Lemma 13.12.1),
- $\mathrm{R}^1\mathrm{Pic}^*(\tilde{X}) = \mathrm{R}^1\mathrm{Pic}^c(\tilde{X}) = [0 \rightarrow \mathrm{Pic}^0(\overline{X}, Y)]$ since \tilde{X} is smooth (Theorem 13.11.1; note that $\mathcal{U} = 0$ by the smoothness of \tilde{X}),

- $R^1\mathrm{Pic}(M(S)^*(n)[2n]) = 0$ (Lemma 13.12.1),
- $R^2\mathrm{Pic}(M(\tilde{S})^*(n)[2n]) = [\mathbb{Z}\pi_0^c(\tilde{S}^{[n-1]}) \rightarrow 0] := [\mathrm{Div}_{\tilde{S}}(\overline{X}, Y) \rightarrow 0]$ is given by the free abelian group on compact irreducible components of \tilde{S} (Lemma 13.12.1),
- $R^2\mathrm{Pic}^*(\tilde{X}) = R^2\mathrm{Pic}^c(\tilde{X}) = R^2\mathrm{Pic}(\overline{X}, Y) = [\mathrm{NS}_c^{(0)}(\tilde{X}) \xrightarrow{u^2} G_c^{(2)}]$ (Theorem 13.11.4),
- $R^2\mathrm{Pic}(M(S)^*(n)[2n]) = [\mathbb{Z}\pi_0^c(S^{[n-1]}) \rightarrow 0] = [\mathrm{Div}_S(X) \rightarrow 0]$ (Lemma 13.12.1).

We may therefore rewrite the above exact sequence as follows:

$$0 \rightarrow [0 \rightarrow \mathrm{Pic}^0(\overline{X}, Y)] \rightarrow R^1\mathrm{Pic}^*(X) \rightarrow [\mathrm{Div}_{\tilde{S}}(\overline{X}, Y) \rightarrow 0] \\ \xrightarrow{\alpha} [\mathrm{NS}_c^{(0)}(\tilde{X}) \xrightarrow{u^2} G_c^{(2)}] \oplus [\mathrm{Div}_S(X) \rightarrow 0].$$

The map $\mathrm{Div}_{\tilde{S}}(\overline{X}, Y) \rightarrow \mathrm{Div}_S(X)$ induced from $M(\tilde{S}) \rightarrow M(S)$ is clearly the proper push-forward of Weil divisors. The map

$$[\mathrm{Div}_{\tilde{S}}(\overline{X}, Y) \rightarrow 0] \rightarrow [\mathrm{NS}_c^{(0)}(\tilde{X}) \xrightarrow{u^2} G_c^{(2)}]$$

is the cycle class map described in Corollary 13.12.5. By Corollary 13.12.3 we then get

$$\mathrm{Ker} \alpha = [\mathrm{Div}_{\tilde{S}/S}^0(\overline{X}, Y) \rightarrow 0]$$

where the lattice $\mathrm{Div}_{\tilde{S}/S}^0(\overline{X}, Y)$ is from the definition of Pic^- (see [8, 2.2.1]). In other terms, we have

$$R^1\mathrm{Pic}^*(X) = [\mathrm{Div}_{\tilde{S}/S}^0(\overline{X}, Y) \xrightarrow{u} \mathrm{Pic}^0(\overline{X}, Y)]$$

and we are left to check that the mapping u is the one described in [8]. Just observe that, by Lemma 13.12.1 and Theorem 13.12.4,

$$R^1\mathrm{Pic}^*(X) \hookrightarrow R^1\mathrm{Pic}^*(X, S) \cong R^1\mathrm{Pic}^*(\tilde{X}, \tilde{S}) \cong R^1\mathrm{Pic}^*(\tilde{X}, Z) \\ = R^1\mathrm{Pic}^*(\overline{X} - Y, Z) \cong R^1\mathrm{Pic}(\overline{X} - Z, Y)$$

and the latter is isomorphic to $\mathrm{Pic}^+(\overline{X} - Z, Y)$ by 13.12.2. Since, by construction, $\mathrm{Pic}^-(X)$ is a sub-1-motive of $\mathrm{Pic}^+(\overline{X} - Z, Y)$ the isomorphism of 13.12.2 restricts to the claimed one. \square

14. GENERALISATIONS OF ROİTMAN'S THEOREM

In this section, we give a unified treatment of Roİtman's theorem on torsion 0-cycles on a smooth projective variety and its various generalisations.

14.1. Motivic and classical Albanese. Let $X \in \text{Sch}(k)$; we assume X smooth if $p > 1$ and X semi-normal (in particular reduced) if $p = 1$, see Lemma 9.4.1. Recall that Suslin's algebraic singular homology is

$$H_j(X) := \text{Hom}_{\text{DM}_{-}^{\text{eff}}}(\mathbb{Z}[j], M(X)) = \mathbb{H}_{\text{Nis}}^{-j}(k, C_*(X))$$

for any scheme $p : X \rightarrow k$. On the other hand, we may define

$$H_j^{\text{ét}}(X) := \text{Hom}_{\text{DM}_{-}^{\text{eff}}, \text{ét}}(\mathbb{Z}[j], M_{\text{ét}}(X)) = \mathbb{H}_{\text{ét}}^{-j}(k, \alpha^* C_*(X)).$$

We also have versions with coefficients in an abelian group A :

$$H_j^{\text{ét}}(X, A) = \text{Hom}_{\text{DM}_{-}^{\text{eff}}, \text{ét}}(\mathbb{Z}[j], M_{\text{ét}}(X) \otimes A).$$

We shall also use the following notation throughout:

14.1.1. Notation. For any $M \in \text{DM}_{\text{gm}}^{\text{eff}}$ and any abelian group A , we write $H_j^{(1)}(M, A)$ for the abelian group

$$\text{Hom}_{\text{DM}_{-}^{\text{eff}}, \text{ét}}(\mathbb{Z}[j], \text{Tot LAlb}(M) \otimes A) \simeq \mathbb{H}_{\text{ét}}^{-j}(k, \text{Tot LAlb}(M) \overset{L}{\otimes} A).$$

This is *Suslin 1-motivic homology of M with coefficients in A* . If $M = M(X)$, we write $H_j^{(1)}(X, A)$ for $H_j^{(1)}(M, A)$. We drop A in the case where $A = \mathbb{Z}$.

We also write $H_j^{\text{ét}}(M, A) = \text{Hom}_{\text{DM}_{-}^{\text{eff}}}(\mathbb{Z}[j], M \otimes A)$ and $H_j^{\text{ét}}(M) = H_j^{\text{ét}}(M, \mathbb{Z})$.

The motivic Albanese map (5.2) then gives maps

$$(14.1) \quad H_j^{\text{ét}}(M, A) \rightarrow H_j^{(1)}(M, A)$$

for any abelian group A .

14.1.2. Proposition. *If X is a smooth curve (or any curve in characteristic 0), the map (14.1) (for $M = M(X)$) is an isomorphism for any A, j .*

Proof. This follows immediately from Proposition 5.1.4. \square

Note that if $X = \overline{X} - Y$ is a smooth curve obtained by removing a finite set of closed points from a projective smooth curve \overline{X} then $\mathcal{A}_{X/k} = \text{Pic}_{(\overline{X}, Y)/k}$ is the relative Picard scheme (see [8] for its representability) and the Albanese map just sends a point $P \in X$ to $(\mathcal{O}_{\overline{X}}(P), 1)$ where 1 is the tautological section, trivialising $\mathcal{O}_{\overline{X}}(P)$ on X . We then have the following result (*cf.* [61, Lect. 7, Th. 7.16]).

14.1.3. Corollary. *If $X = \overline{X} - Y$ is a smooth curve,*

$$H_0^{\text{ét}}(X) \rightarrow \text{Pic}(\overline{X}, Y)[1/p]$$

is an isomorphism.

Now let $\mathcal{A}_{X/k}^{\text{éh}}$ be as in Proposition 13.6.1 and Remark 13.6.2. The map $\text{Tot LAlb}(X) \rightarrow \mathcal{A}_{X/k}^{\text{éh}}$ of *loc. cit.* induces a homomorphism

$$(14.2) \quad H_0^{(1)}(X) \rightarrow \mathcal{A}_{X/k}^{\text{éh}}(k)[1/p]$$

which is not an isomorphism in general (but see Lemma 14.4.2). Composing (14.2), (14.1) (for $A = \mathbb{Z}$) and the obvious map $H_0(X)[1/p] \rightarrow H_0^{\text{ét}}(X)$, we get a map

$$(14.3) \quad H_0(X)[1/p] \rightarrow \mathcal{A}_{X/k}^{\text{éh}}(k)[1/p].$$

We may further restrict to parts of degree 0, getting a map

$$H_0(X)^0[1/p] \rightarrow (\mathcal{A}_{X/k}^{\text{éh}})^0(k)[1/p].$$

Recall that $\mathcal{A}_{X/k}^{\text{éh}} = \mathcal{A}_{X/k}$ if X is normal or strictly reduced (Proposition 13.6.1 d)). In this case, the above map becomes

$$(14.4) \quad H_0(X)^0[1/p] \rightarrow (\mathcal{A}_{X/k})^0(k)[1/p].$$

If X is smooth, (14.4) is the $\mathbb{Z}[1/p]$ -localisation of the generalised Albanese map of Spieß-Szamuely [81, (2)].

Dually to Lemma C.6.2, the functor

$$D^b(\mathcal{M}_1[1/p]) \rightarrow \text{DM}_{-, \text{ét}}^{\text{eff}} \\ C \mapsto \text{Tot}(C) \otimes (\mathbb{Q}/\mathbb{Z})'$$

is exact with respect to the ${}_t\mathcal{M}_1$ t -structure on the left and the homotopy t -structure on the right; here, as usual, $(\mathbb{Q}/\mathbb{Z})' := \bigoplus_{l \neq p} \mathbb{Q}_l/\mathbb{Z}_l$. In other words:

14.1.4. Lemma. *For any $C \in D^b(\mathcal{M}_1[1/p])$, there are canonical isomorphisms of sheaves*

$$\mathcal{H}_j(\text{Tot}(C) \otimes (\mathbb{Q}/\mathbb{Z})') \simeq \text{Tot}({}_tH_j(C)) \otimes (\mathbb{Q}/\mathbb{Z})'$$

(note that the right hand side is a single sheaf!) In particular, for $C = \text{LAlb}(M)$ and k algebraically closed:

$$H_j^{(1)}(M, (\mathbb{Q}/\mathbb{Z})') \simeq \Gamma(k, \text{Tot}(\text{L}_j \text{Alb}(M)) \otimes (\mathbb{Q}/\mathbb{Z})').$$

14.2. A variant of the Suslin-Voevodsky theorem. *We now assume that k is algebraically closed until the end of this section.*

Let n be invertible in k . For $M \in \text{DM}_{\text{gm}, \text{ét}}^{\text{eff}}$, we have the composition pairing

$$(14.5) \quad \text{Hom}(\mathbb{Z}/n, M) \times \text{Hom}(M, \mathbb{Z}/n) \rightarrow \text{Hom}(\mathbb{Z}/n, \mathbb{Z}/n) = \mathbb{Z}/n.$$

14.2.1. Theorem. *For any $M \in \text{DM}_{\text{gm}, \text{ét}}^{\text{eff}}$, the pairing (14.5) is a perfect duality of finite \mathbb{Z}/n -modules.*

Proof. The statement is stable under exact triangles and direct summands, thus it is enough to check it for $M = M(X)[-j]$, X smooth, $j \in \mathbb{Z}$. Then the statement amounts to the duality between the group $\mathrm{Hom}(M(X), \mathbb{Z}/n[j])$ and algebraic singular homology $H_j^{\mathrm{sing}}(X, \mathbb{Z}/n) = \mathrm{Hom}(\mathbb{Z}/n[j], M(X))$, which is the contents of [61, Th. 10.9]. \square

14.3. Change of topology and motivic Albanese map. We have the change of topology functor

$$\alpha^* : \mathrm{DM}_{\mathrm{gm}}^{\mathrm{eff}} \rightarrow \mathrm{DM}_{\mathrm{gm}, \mathrm{ét}}^{\mathrm{eff}}.$$

Recall the functor $d_{\leq 1}$ of (5.1) and the motivic Albanese map a_M of (5.2). Note that $a_{\mathbb{Z}/n(1)} : \alpha^* \mathbb{Z}/n(1) \rightarrow d_{\leq 1} \mathbb{Z}/n(1)$ is an isomorphism by Proposition 5.1.4. This gives a meaning to:

14.3.1. Proposition. *Let $M \in \mathrm{DM}_{\mathrm{gm}, \mathrm{ét}}^{\mathrm{eff}}$. Then:*

a) *The diagram*

$$\begin{array}{ccc} \mathrm{Hom}_{\mathrm{DM}_{\mathrm{gm}}^{\mathrm{eff}}}(M, \mathbb{Z}/n(1)) & \xrightarrow{d_{\leq 1}} & \mathrm{Hom}_{\mathrm{DM}_{\mathrm{gm}, \mathrm{ét}}^{\mathrm{eff}}}(d_{\leq 1} M, \alpha^* \mathbb{Z}/n(1)) \\ & \searrow \alpha^* & \downarrow (a_M)^* \\ & & \mathrm{Hom}_{\mathrm{DM}_{\mathrm{gm}, \mathrm{ét}}^{\mathrm{eff}}}(\alpha^* M, \alpha^* \mathbb{Z}/n(1)) \end{array}$$

commutes.

b) *In this diagram, $d_{\leq 1}$ is an isomorphism.*

Proof. Let $N \in \mathrm{DM}_{\mathrm{gm}, \mathrm{ét}}^{\mathrm{eff}}$. By the naturality of the motivic Albanese map, the diagram

$$\begin{array}{ccc} \mathrm{Hom}_{\mathrm{Nis}}(M, N) & \xrightarrow{d_{\leq 1}} & \mathrm{Hom}_{\mathrm{ét}}(d_{\leq 1} M, d_{\leq 1} N) \\ \alpha^* \downarrow & & \downarrow (a_M)^* \\ \mathrm{Hom}_{\mathrm{ét}}(\alpha^* M, \alpha^* N) & \xrightarrow{(a_N)^*} & \mathrm{Hom}_{\mathrm{ét}}(\alpha^* M, d_{\leq 1} N) \end{array}$$

commutes. Taking $N = \mathbb{Z}/n(1)$, we get a).

For b), we write $d_{\leq 1}$ as a composition $D_{\leq 1}^{\mathrm{ét}} \circ \alpha^* \circ D_{\leq 1}^{\mathrm{Nis}}$. We shall show that each of these three functors induces an isomorphism on the corresponding Hom groups.

For $D_{\leq 1}^{\mathrm{Nis}}$, this is because the map

$$D_{\leq 1}^{\mathrm{Nis}} : \mathrm{Hom}_{\mathrm{Nis}}(M, \mathbb{Z}/n(1)) \rightarrow \mathrm{Hom}_{\mathrm{Nis}}(\mathbb{Z}/n, D_{\leq 1}^{\mathrm{Nis}}(M))$$

coincides with the adjunction isomorphism for $\underline{\mathrm{Hom}}_{\mathrm{Nis}}$. For α^* , this is because k is algebraically closed, so that étale cohomology of $\mathrm{Spec} k$ coincides with Nisnevich cohomology. Finally, for $D_{\leq 1}^{\mathrm{ét}}$, this is because

\mathbb{Z}/n and $\alpha^* D_{\leq 1}^{\text{Nis}}(M)$ are in $d_{\leq 1} \text{DM}_{\text{gm}, \text{ét}}^{\text{eff}}$ (Lemma 5.1.2), and $D_{\leq 1}^{\text{ét}}$ restricts to a perfect duality on this subcategory (Prop. 4.5.1). \square

14.4. A proof of Roïtman's and Spieß-Szamuely's theorems. In this subsection, we only deal with smooth schemes and the characteristic is arbitrary: we shall show how the results of Section 10 allow us to recover the classical theorem of Roïtman on torsion 0-cycles up to p -torsion, as well as its generalisation to smooth varieties by Spieß-Szamuely [81]. The reader should compare our argument with theirs (*loc. cit.*, §5).

Since k is algebraically closed, Corollary D.1.6 implies

14.4.1. Lemma. *For any $j \in \mathbb{Z}$, $H_j^{\text{ét}}(X) = H_j(X)[1/p]$; similarly with finite or divisible coefficients.* \square

Moreover, it is easy to evaluate $H_j^{(1)}(X) = \mathcal{H}_j(\text{Tot LAlb}(X))(k)$ out of Theorem 10.2.2: if $L_n \text{Alb}(X) = [L_n \rightarrow G_n]$, we have a long exact sequence coming from Proposition 3.10.2

$$(14.6) \quad L_{j+1}(k)[1/p] \rightarrow G_{j+1}(k)[1/p] \rightarrow H_j^{(1)}(X) \\ \rightarrow L_j(k)[1/p] \rightarrow G_j(k)[1/p] \rightarrow \dots$$

Thus:

14.4.2. Lemma. *For X smooth, (14.2) is an isomorphism and we have*

$$(14.7) \quad H_1^{(1)}(X) \simeq \text{NS}_{X/k}^*(k)[1/p] \\ H_j^{(1)}(X) = 0 \text{ if } j \neq 0, 1.$$

Here is now the main lemma:

14.4.3. Lemma. *Let $M = M(X)$ with X smooth, and let $A = \mathbb{Z}/n$ with $(n, p) = 1$. Then the map (14.1) is an isomorphism for $j = 0, 1$ and surjective for $j = 2$.*

Proof. By theorem 14.2.1, the statement is equivalent to the following: the motivic Albanese map

$$(a_X)^* : \text{Hom}_{\text{ét}}(d_{\leq 1} M(X), \alpha^* \mathbb{Z}/n(1)[j]) \rightarrow \text{Hom}_{\text{ét}}(\alpha^* M(X), \alpha^* \mathbb{Z}/n(1)[j])$$

induced by (8.3) is bijective for $j = 0, 1$ and injective for $j = 2$. (Here we also use the fact that $\alpha^* \mathbb{Z}/n(1) = \mu_n$ and that k is algebraically closed.)

By Proposition 14.3.1, we may replace the above map by the change of topology map

$$\begin{aligned} \alpha^* : H_{\text{Nis}}^j(X, \mathbb{Z}/n(1)) &= \text{Hom}_{\text{Nis}}(M(X), \mathbb{Z}/n(1)[j]) \\ &\rightarrow \text{Hom}(\alpha^* M(X), \alpha^* \mathbb{Z}/n(1)[j]) = H_{\text{et}}^j(X, \mathbb{Z}/n(1)). \end{aligned}$$

Then the result follows from Hilbert's theorem 90 (aka Beilinson-Lichtenbaum in weight 1). \square

From this and Lemma 14.1.4 we deduce:

14.4.4. Corollary. *The homomorphism (14.1)*

$$H_j(X, (\mathbb{Q}/\mathbb{Z})') \rightarrow H_j^{(1)}(X, (\mathbb{Q}/\mathbb{Z})')$$

(see Lemma 14.4.1) is bijective for $j = 0, 1$ and surjective for $j = 2$.

The following theorem extends in particular [81, Th. 1.1] to all smooth varieties¹³.

14.4.5. Theorem. *a) The maps (14.3) and (14.4) are isomorphisms on torsion.*

b) $H_1(X) \otimes (\mathbb{Q}/\mathbb{Z})' = 0$.

c) The map (14.1) for $A = \mathbb{Z}$ yields a surjection

$$H_1(X)\{p'\} \twoheadrightarrow \text{NS}_{X/k}^*(k)\{p'\}$$

where $M\{p'\}$ denotes the torsion prime to p in an abelian group M .

Proof. Lemmas 14.4.1 and 14.4.2 reduce us to show that (14.1) is an isomorphism on torsion for $A = \mathbb{Z}$, $j = 0$. We have commutative diagrams with exact rows:

$$(14.8) \quad \begin{array}{ccccccc} 0 \rightarrow & H_j(X) \otimes (\mathbb{Q}/\mathbb{Z})' & \rightarrow & H_j(X, (\mathbb{Q}/\mathbb{Z})') & \rightarrow & H_{j-1}(X)\{p'\} & \rightarrow 0 \\ & \downarrow & & \downarrow & & \downarrow & \\ 0 \rightarrow & H_j^{(1)}(X) \otimes (\mathbb{Q}/\mathbb{Z})' & \rightarrow & H_j^{(1)}(X, (\mathbb{Q}/\mathbb{Z})') & \rightarrow & H_{j-1}^{(1)}(X)\{p'\} & \rightarrow 0 \end{array}$$

For $j = 1$, the middle vertical map is an isomorphism by Lemma 14.4.3 or Corollary 14.4.4 and $H_1^{(1)}(X) \otimes \mathbb{Q}/\mathbb{Z} = 0$ by (14.7), which gives a) and b). For $j = 2$, the middle map is surjective by the same lemma and corollary, which gives c). The proof is complete. \square

¹³In *loc. cit.*, X is supposed to admit an open embedding into a smooth projective variety.

14.4.6. Remark. If X is smooth projective of dimension n , $H_j(X)$ is isomorphic to the higher Chow group $CH^n(X, j)$. In (14.8) for $j = 2$, the lower left term is 0 by Lemma 14.4.2. The composite map

$$H_2(X, (\mathbb{Q}/\mathbb{Z})') \rightarrow H_1(X)\{p'\} \rightarrow H_1^{(1)}(X)\{p'\} = \mathrm{NS}_{X/k}^*(k)\{p'\}$$

is “dual” to the map

$$\mathrm{NS}(X) \otimes (\mathbb{Q}/\mathbb{Z})' \rightarrow H_{\mathrm{\acute{e}t}}^2(X, \mathbb{Q}/\mathbb{Z}(1))$$

whose cokernel is $Br(X)\{p'\}$. Let

$$Br(X)^D = \varinjlim_{(n,p)=1} \mathrm{Hom}({}_n Br(X), \mu_n) :$$

a diagram chase in (14.8) for $j = 2$ then yields an exact sequence

$$\begin{aligned} 0 \rightarrow CH^n(X, 2) \otimes (\mathbb{Q}/\mathbb{Z})' &\rightarrow Br(X)^D \\ &\rightarrow CH^n(X, 1)\{p'\} \rightarrow \mathrm{NS}_{X/k}^*(k)\{p'\} \rightarrow 0. \end{aligned}$$

Together with $CH^n(X, 1) \otimes (\mathbb{Q}/\mathbb{Z})' = 0$, this should be considered as a natural complement to Roitman’s theorem.

14.5. Generalisation to singular schemes. We now assume $\mathrm{char} k = 0$, and show how the results of Section 11 allow us to extend the results of the previous subsection to singular schemes. By blow-up induction and the 5 lemma, we get:

14.5.1. Proposition. *The isomorphisms and surjection of Lemma 14.4.3 and Corollary 14.4.4 extend to all $X \in \mathrm{Sch}$. \square*

Let $L_1 \mathrm{Alb}(X) = [L_1 \xrightarrow{u_1} G_1]$. Proposition 14.5.1, the exact sequence (14.6) and the snake chase in the proof of Theorem 14.4.5 give:

14.5.2. Corollary. *For $X \in \mathrm{Sch}$, we have exact sequences*

$$\begin{aligned} 0 \rightarrow H_1(X) \otimes \mathbb{Q}/\mathbb{Z} &\rightarrow \mathrm{Ker}(u_1) \otimes \mathbb{Q}/\mathbb{Z} \rightarrow H_0(X)_{\mathrm{tors}} \rightarrow \mathrm{Coker}(u_1)_{\mathrm{tors}} \rightarrow 0 \\ 0 \rightarrow H_1(X) \otimes \mathbb{Q}/\mathbb{Z} &\rightarrow H_1^{(1)}(X, \mathbb{Q}/\mathbb{Z}) \rightarrow H_0(X)_{\mathrm{tors}} \rightarrow 0. \end{aligned}$$

The second exact sequence is more intrinsic than the first, but note that it does not give information on $H_1(X) \otimes \mathbb{Q}/\mathbb{Z}$.

14.5.3. Corollary. *If X is normal, $H_1(X) \otimes \mathbb{Q}/\mathbb{Z} = 0$ and there is an isomorphism*

$$\mathcal{A}_{X/k}(k)_{\mathrm{tors}} \xrightarrow{\sim} H_0(X)_{\mathrm{tors}}.$$

Proof. This follows from the previous corollary and Corollary 13.6.6 c). \square

14.5.4. Remark. Theorem 13.10.1 shows that the second isomorphism of Proposition 14.5.1 coincides with the one of Geisser in [37, Th. 6.2] when X is proper. When X is further normal, the isomorphism of Corollary 14.5.3 also coincides with the one of his Theorem 6.1.

14.5.5. Remarks. Note that the reformulation of “Roitman’s theorem” involving $\text{Ker } u_1$ is the best possible!

1) Let X be a proper scheme such that $\text{Pic}^0(X)/\mathcal{U} = \mathbb{G}_m^r$ is a torus (more likely such that $\text{Alb}(X_0) = 0$ where $X_0 \rightarrow X$ is a resolution, according with the description in [8, p. 68]). Then $R^1\text{Pic}(X)^* = L_1\text{Alb}(X) = [\mathbb{Z}^r \rightarrow 0]$ is the character group (cf. [8, 5.1.4]). For example, take a nodal projective curve X with resolution $X_0 = \mathbb{P}^1$. In this case the map (14.1) for $A = \mathbb{Z}$ is an isomorphism for all j and thus $\text{Ker}(u_1) \otimes \mathbb{Q}/\mathbb{Z} = H_1(X) \otimes \mathbb{Q}/\mathbb{Z} = (\mathbb{Q}/\mathbb{Z})^r$.

2) For Borel-Moore and $L_1\text{Alb}^c(X) = L_1\text{Alb}^*(X)$ for X smooth open is Cartier dual of $\text{Pic}^0(\overline{X}, Y)$ then (cf. [8, p. 47]) $\text{Ker } u_1^c$ can be non-zero: take $\overline{X} = \mathbb{P}^1$ and $Y =$ a finite number of points.

14.6. Borel-Moore Roitman. We are still in characteristic 0. Recall that the Borel-Moore motivic homology group

$$H_j^c(X, \mathbb{Z}) := \text{Hom}(\mathbb{Z}[j], M^c(X))$$

is canonically isomorphic to Bloch’s higher Chow group $CH_0(X, j)$. Similarly to the previous sections, we have maps

$$\begin{aligned} H_j^c(X, \mathbb{Z}) &\rightarrow H_j^{(1)}(M^c(X)) =: H_j^{c,(1)}(X) \\ H_j^c(X, \mathbb{Q}/\mathbb{Z}) &\rightarrow H_j^{(1)}(M^c(X), \mathbb{Q}/\mathbb{Z}) =: H_j^{c,(1)}(X, \mathbb{Q}/\mathbb{Z}) \end{aligned}$$

and

14.6.1. Proposition. *The second map is an isomorphism for $j = 0, 1$ and surjective for $j = 2$.*

Proof. By localisation induction, reduce to X proper and use Proposition 14.5.1. \square

14.6.2. Corollary. *For $X \in \text{Sch}$, we have exact sequences*

$$\begin{aligned} 0 \rightarrow CH_0(X, 1) \otimes \mathbb{Q}/\mathbb{Z} &\rightarrow \text{Ker}(u_1^c) \otimes \mathbb{Q}/\mathbb{Z} \\ &\rightarrow CH_0(X)_{\text{tors}} \rightarrow \text{Coker}(u_1^c)_{\text{tors}} \rightarrow 0 \\ 0 \rightarrow CH_0(X, 1) \otimes \mathbb{Q}/\mathbb{Z} &\rightarrow H_j^{c,(1)}(X, \mathbb{Q}/\mathbb{Z}) \rightarrow CH_0(X)_{\text{tors}} \rightarrow 0 \end{aligned}$$

where we write $L_1\text{Alb}^c(X) = [L_1^c \xrightarrow{u_1^c} G_1^c]$. In particular, if X is smooth quasi-affine of dimension > 1 , $G_1^c = 0$.

Proof. Only the last assertion needs a proof: if X is smooth affine of dimension > 1 then $CH_0(X)_{\text{tors}} = 0$ [23, Th. 4.1 (iii)], hence $\text{Coker}(u_1^c)_{\text{tors}} = 0$; this forces the semi-abelian variety G_1^c to be 0. We may then pass from affine to quasi-affine by using the localisation exact sequence and the description of $L_0\text{Alb}^c$ in Proposition 11.5.2 b). \square

14.7. “Cohomological” Roïtman. We are still in characteristic 0.

14.7.1. Lemma. *Let $0 < r \leq n$. Then for any $Z \in \text{Sch}$ of dimension $\leq n - r$ and any $i > 2(n - r)$, we have $H_{\text{cdh}}^i(Z, \mathbb{Q}/\mathbb{Z}(n)) = 0$.*

Proof. By blow-up induction we reduce to the case where Z is smooth of pure dimension $n - r$; then $H_{\text{cdh}}^i(Z, \mathbb{Q}/\mathbb{Z}(n)) = H_{\text{Nis}}^i(Z, \mathbb{Q}/\mathbb{Z}(n))$. Since k is algebraically closed, and $n \geq \dim Z$, $H_{\text{Nis}}^i(Z, \mathbb{Q}/\mathbb{Z}(n)) \simeq H_{\text{ét}}^i(Z, \mathbb{Q}/\mathbb{Z}(n))$ by Suslin’s theorem [82] and the vanishing follows from the known bound for étale cohomological dimension. \square

Now consider the 1-motive $L_1\text{Alb}^*(X)$ for X of dimension n . This time, we have maps

$$(14.9) \quad \begin{aligned} H_{\text{cdh}}^{2n-j}(X, \mathbb{Z}(n)) &\rightarrow H_j^{(1)}(M(X)^*(n)[2n]) =: H_{(1)}^{2n-j}(X, \mathbb{Z}(n)) \\ H_{\text{cdh}}^{2n-j}(X, \mathbb{Q}/\mathbb{Z}(n)) &\rightarrow H_j^{(1)}(M(X)^*(n)[2n], \mathbb{Q}/\mathbb{Z}) =: H_{(1)}^{2n-j}(X, \mathbb{Q}/\mathbb{Z}(n)). \end{aligned}$$

14.7.2. Lemma. *Let $Z \in \text{Sch}$ be of dimension $< n$. Then the map*

$$H_{\text{cdh}}^{2n-2}(Z, \mathbb{Q}/\mathbb{Z}(n)) \rightarrow H_2^{(1)}(M(Z)^*(n)[2n], \mathbb{Q}/\mathbb{Z})$$

is an isomorphism.

Proof. For notational simplicity, write $H^*(Y, n)$ for $H_{\text{cdh}}^*(Y, \mathbb{Q}/\mathbb{Z}(n))$ and $F_j(Y)$ for $H_j^{(1)}(M(Y)^*(n)[2n], \mathbb{Q}/\mathbb{Z})$, where Y is a scheme of dimension $\leq n$. Let \tilde{Z}, T, \tilde{T} be as in the proof of Lemma 13.12.1. Then Lemma 14.7.1 and proposition 8.1.2 yield a commutative diagram

$$\begin{array}{ccc} H^{2n-2}(Z, n) & \longrightarrow & H^{2n-2}(\tilde{Z}, n) \\ \downarrow & & \downarrow \\ F_2(Z) & \longrightarrow & F_2(\tilde{Z}) \end{array}$$

in which both horizontal maps are isomorphisms. Therefore, it suffices to prove the lemma when Z is smooth quasiprojective of dimension $n - 1$.

The motive $\text{R}^2\text{Pic}(M(Z)^*(n)[2n]) \simeq \text{R}^2\text{Pic}(M^c(Z)(1)[2])$ was computed in Corollary 13.11.5: it is $[\mathbb{Z}^{\pi_0^c(Z)} \rightarrow 0]$. Therefore, we get

$$F_2(Z) \simeq \mathbb{Q}/\mathbb{Z}(1)[\pi_0^c(Z)].$$

On the other hand, the trace map defines an isomorphism

$$H^{2n-2}(Z, n) \xrightarrow{\sim} \mathbb{Q}/\mathbb{Z}(1)[\pi_0^c(Z)]$$

and the issue is to prove that the vertical map in the diagram is this isomorphism. For this, we first may reduce to Z projective and connected. Now we propose the following argument: take a chain of smooth closed subvarieties $Z \supset Z_2 \supset \cdots \supset Z_n$, with Z_i of dimension $n - i$ and connected (take multiple hyperplane sections up to Z_{n-1} and then a single point of Z_{n-1} for Z_n). The Gysin exact triangles give commutative diagrams

$$\begin{array}{ccc} H^{2n-2i-2}(Z_{i+1}, n-i) & \longrightarrow & H^{2n-2i}(Z_i, n-i+1) \\ \downarrow & & \downarrow \\ F_2(Z_{i+1}) & \longrightarrow & F_2(Z_i) \end{array}$$

in which both horizontal maps are isomorphisms: thus we are reduced to the case $\dim Z = 0$, where it follows from Proposition 14.1.2 applied to $X = \mathbb{P}^1$. \square

14.7.3. Theorem. *The map (14.9) is an isomorphism for $j = 0, 1$.*

Proof. This is easy and left to the reader for $j = 0$. For $j = 1$, we argue as usual by blowup induction. In the situation of 8.1.e, we then have a commutative diagram of exact sequences

$$\begin{array}{ccccccc} H^{2n-2}(\tilde{X}, n) \oplus H^{2n-2}(Z, n) & \rightarrow & H^{2n-2}(\tilde{Z}, n) & \rightarrow & H^{2n-1}(X, n) & \rightarrow & H^{2n-1}(\tilde{X}, n) \oplus H^{2n-1}(Z, n) \\ \downarrow & & \downarrow & & \downarrow & & \downarrow \\ F_2(\tilde{X}) \oplus F_2(Z) & \rightarrow & F_2(\tilde{Z}) & \rightarrow & F_1(X) & \rightarrow & F_1(\tilde{X}) \oplus F_1(Z). \end{array}$$

In this diagram, we have $F_1(Z) = 0$ by Lemma 13.12.1 and $H^{2n-1}(Z, n) = 0$ by Lemma 14.7.1, and the same lemmas imply that both rightmost horizontal maps are surjective. The rightmost vertical map is now an isomorphism by Proposition 14.6.1, which also gives the surjectivity of $H^{2n-2}(\tilde{X}, n) \rightarrow F_2(\tilde{X})$. Finally, Lemma 14.7.2 implies that $H^{2n-2}(\tilde{Z}, n) \rightarrow F_2(\tilde{Z})$ and $H^{2n-2}(Z, n) \rightarrow F_2(Z)$ are isomorphisms, and the conclusion follows from the 5 lemma. \square

14.7.4. Corollary. *For $X \in \text{Sch}$ of dimension n , we have exact sequences*

$$\begin{aligned} 0 \rightarrow H^{2n-1}(X, \mathbb{Z}(n)) \otimes \mathbb{Q}/\mathbb{Z} &\rightarrow \text{Ker}(u_1^*) \otimes \mathbb{Q}/\mathbb{Z} \\ &\rightarrow H^{2n}(X, \mathbb{Z}(n))_{\text{tors}} \rightarrow \text{Coker}(u_1^*)_{\text{tors}} \rightarrow 0 \end{aligned}$$

$$0 \rightarrow H^{2n-1}(X, \mathbb{Z}(n)) \otimes \mathbb{Q}/\mathbb{Z} \rightarrow H_{(1)}^{2n-1}(X, \mathbb{Q}/\mathbb{Z}(n)) \rightarrow H^{2n}(X, \mathbb{Z}(n))_{\text{tors}} \rightarrow 0$$

where u_1^* is the map involved in the 1-motive $L_1\text{Alb}^*(X)$ (which is isomorphic to $\text{Alb}^+(X)$ by the dual of Theorem 13.12.6).

14.7.5. Corollary. *If X is a proper scheme of dimension n we then get $H^{2n-1}(X, \mathbb{Z}(n)) \otimes \mathbb{Q}/\mathbb{Z} = 0$ and an isomorphism*

$$\text{Alb}^+(X)(k)_{\text{tors}} \xrightarrow{\sim} H^{2n}(X, \mathbb{Z}(n))_{\text{tors}}.$$

Proof. If X is proper then $L_1\text{Alb}^*(X) \cong \text{Alb}^+(X)$ is semiabelian and the claim follows from the previous corollary. \square

14.7.6. Remark. Marc Levine outlined us how to construct a “cycle map” $c\ell^{\text{cdh}}$ from $CH_{LW}^n(X)$ to $H_{\text{cdh}}^{2n}(X, \mathbb{Z}(n))$, where $CH_{LW}^n(X)$ is the Levine-Weibel cohomological Chow group of zero cycles. This gives a map

$$c\ell_{\text{tors}}^{\text{cdh}} : CH_{LW}^n(X)_{\text{tors}} \rightarrow H_{\text{cdh}}^{2n}(X, \mathbb{Z}(n))_{\text{tors}}$$

which most likely fits in a commutative diagram (for X projective)

$$\begin{array}{ccc} CH_{LW}^n(X)_{\text{tors}} & \xrightarrow{c\ell_{\text{tors}}^{\text{cdh}}} & H_{\text{cdh}}^{2n}(X, \mathbb{Z}(n))_{\text{tors}} \\ a_{\text{tors}}^+ \downarrow & & \downarrow \wr \\ \text{Alb}^+(X)(k)_{\text{tors}} & \xrightarrow{\sim} & L_1\text{Alb}^*(X)(k)_{\text{tors}} \end{array}$$

where: the horizontal bottom isomorphism is that induced by Theorem 13.12.6 and the right vertical one comes from the previous Corollary 14.7.4; the left vertical map is the one induced, on torsion, by the universal regular homomorphism $a^+ : CH_{LW}^n(X)_{\text{deg } 0} \rightarrow \text{Alb}^+(X)(k)$ constructed in [8, 6.4.1]. This would imply that

$$c\ell_{\text{tors}}^{\text{cdh}} \text{ is an isomorphism } \iff a_{\text{tors}}^+ \text{ is an isomorphism.}$$

If X is normal and $k = \bar{k}$ or for any X projective if $k = \mathbb{C}$ then a_{tors}^+ is known to be an isomorphism, see [53]. For X projective over any algebraically closed field, see Mallick [57].

We expect that Levine’s “cycle map” $c\ell^{\text{cdh}}$ is surjective with uniquely divisible kernel (probably representable by a unipotent group).

Part 4. Realisations

15. AN AXIOMATIC VERSION OF DELIGNE’S CONJECTURE

Let k be a perfect field. We shall drop the reference to k from the notation for categories of motives associated to k .

15.1. A review of base change. Suppose given a diagram of categories and functors

$$(15.1) \quad \begin{array}{ccc} \mathcal{T}_1 & \xrightleftharpoons[A]{T} & \mathcal{T} \\ R_1 \uparrow & & \uparrow R \\ \mathcal{D}_1 & \xrightleftharpoons[B]{S} & \mathcal{D} \end{array}$$

where A is left adjoint to T and B is left adjoint to S , plus a natural transformation $\varphi : RS \Rightarrow TR_1$. Then we get a natural transformation

$$\psi : AR \Rightarrow R_1B$$

as the adjoint of the composition

$$R \Rightarrow RSB \Rightarrow TR_1B$$

where the first natural transformation is given by the unit $Id_{\mathcal{D}} \Rightarrow SB$ and the second one is induced by φ .

Suppose now that all categories and functors are triangulated. To prove that ψ is an isomorphism of functors, it suffices to check it on generators of \mathcal{D} .

15.2. A weight filtration on $\mathcal{M}_1 \otimes \mathbb{Q}$. In this subsection, we prove that the weight filtration on 1-motives defines a weight filtration on $\mathcal{M}_1 \otimes \mathbb{Q}$ in the sense of Definition E.7.2.

For $w \in \mathbb{Z}$, let $(\mathcal{M}_1 \otimes \mathbb{Q})_{\leq w}$ be the full subcategory consisting of 1-motives of weight $\leq w$ (cf. [25, (10.1.4)] and §C.11). Then $(\mathcal{M}_1 \otimes \mathbb{Q})_{\leq w} = 0$ for $w < -2$ and $(\mathcal{M}_1 \otimes \mathbb{Q})_{\leq w} = \mathcal{M}_1 \otimes \mathbb{Q}$ for $w \geq 0$.

15.2.1. Proposition. *The inclusion functors*

$$\iota_w : (\mathcal{M}_1 \otimes \mathbb{Q})_{\leq w} \hookrightarrow (\mathcal{M}_1 \otimes \mathbb{Q})_{\leq w+1}$$

define a weight filtration on $\mathcal{M}_1 \otimes \mathbb{Q}$.

Proof. By Remark E.7.8, it suffices to check that the weight filtration on 1-motives verifies the conditions in [46, p. 83, Def. 6.3 a)]. The only point is its exactness, which is clear. \square

15.3. A left adjoint in the category of realisations. Let K be a field and \mathcal{T} be a K -linear triangulated category. We assume that \mathcal{T} is provided with a K -linear t -structure, with heart \mathcal{B} with a weight filtration $\mathcal{B}_{\leq n}$ (in the sense of Definition E.7.2) For convenience, we assume that $\mathcal{B}_{\leq 0} = \mathcal{B}$.

Let us consider a thick K -subcategory $\mathcal{B}_{\mathbb{L}}$ of \mathcal{B}_{-2} whose objects are called the *Lefschetz objects*. We assume $\mathcal{B}_{\mathbb{L}}$ semi-simple.

15.3.1. Definition. An object $H \in \mathcal{B}$ is of *level* ≤ 1 if

- (i) The weights of H belong to $\{-2, -1, 0\}$;
- (ii) H_{-2} is a Lefschetz object.

We write $\mathcal{B}_{(1)}$ for the full subcategory of \mathcal{B} consisting of objects of level ≤ 1 .

15.3.2. Proposition.

- (1) $\mathcal{B}_{(1)}$ is a thick abelian subcategory of \mathcal{B} .
- (2) If \mathcal{B}_{-2} is semi-simple, the inclusion functor $\mathcal{B}_{(1)} \hookrightarrow \mathcal{B}$ has an exact left adjoint $H \mapsto \text{Alb}^{\mathcal{B}}(H)$.

Proof. (1) By semi-simplicity of $\mathcal{B}_{\mathbb{L}}$ the assertion is clear.

(2) We shall construct $\text{Alb}^{\mathcal{B}}$ as the composition of two exact functors:

- The first functor sends an object H to $H_{>-3}$.
- Suppose that $H_{\leq -3} = 0$, and consider H_{-2} . By semi-simplicity, we may write $H_{-2} = H' \oplus H''$ with H' a Lefschetz object and with $\text{Hom}_{\mathcal{B}}(L, H'') = \text{Hom}_{\mathcal{B}}(H'', L) = 0$ for any Lefschetz object L . Then the second functor sends H to H/H'' .

The fact that this indeed defines a left adjoint is readily checked. \square

15.3.3. Remark. Let $\mathcal{B}'_{-2} = \{B \in \mathcal{B}_{-2} \mid \mathcal{B}(B, \mathcal{B}_{\mathbb{L}}) = 0\}$. If \mathcal{B}_{-2} is semi-simple, every object $H \in \mathcal{B}_{-2}$ has a unique decomposition $H = H_{\mathbb{L}} \oplus H''$ with $H_{\mathbb{L}} \in \mathcal{B}_{\mathbb{L}}$ and $H'' \in \mathcal{B}'_{-2}$. Let now

$$\mathcal{B}'_{\leq -2} = \{H \in \mathcal{B}_{\leq -2} \mid (H_{-2})_{\mathbb{L}} = 0\}.$$

Then Proposition 15.3.2 yields a split exact sequence

$$(15.2) \quad 0 \rightarrow \mathcal{B}'_{\leq -2} \xrightarrow{i} \mathcal{B} \xrightarrow{\text{Alb}^{\mathcal{B}}} \mathcal{B}_{(1)} \rightarrow 0$$

in which the right adjoint of $\text{Alb}^{\mathcal{B}}$ is the natural inclusion.

15.3.4. Proposition. Let $\mathcal{T}_{(1)}$ be the full subcategory of \mathcal{T} consisting of objects T such that $H^i(T) \in \mathcal{B}_{(1)}$ for all $i \in \mathbb{Z}$. Then:

- (1) $\mathcal{T}_{(1)}$ is a thick subcategory of \mathcal{T} , and the t -structure of \mathcal{T} induces a t -structure on $\mathcal{T}_{(1)}$.
- (2) If the t -structure is bounded and \mathcal{B}_{-2} is semi-simple, the inclusion functor $\mathcal{T}_{(1)} \hookrightarrow \mathcal{T}$ has a t -exact left adjoint $\text{LAlb}^{\mathcal{T}}$.

Proof. (1) follows from Proposition 15.3.2 (1) and (2), since $\mathcal{B}_{(1)}$ is thick in \mathcal{B} by Proposition 15.3.2 (1). For (2), the exact sequence (15.2) of Remark 15.3.3 yields an exact sequence of triangulated categories

$$0 \rightarrow \mathcal{T}'_{\leq -2} \xrightarrow{i} \mathcal{T} \xrightarrow{\pi} \mathcal{T}/\mathcal{T}'_{\leq -2} \rightarrow 0$$

where $\mathcal{T}'_{\leq -2} = \{C \in \mathcal{T} \mid H^*(C) \in \mathcal{B}'_{\leq -2}\}$. The claim now follows from Proposition E.20.1 (3). More precisely, this proposition shows that π

has a right adjoint j such that $j(\mathcal{T}/\mathcal{T}'_{\leq 2}) = \mathcal{T}_{(1)}$; then π gets identified with the desired functor $\mathrm{LAlb}^{\mathcal{T}}$. \square

15.4. Realisation functor. Let $K, \mathcal{T}, \mathcal{B}$ be as in 15.3. For $T \in \mathcal{T}$, we denote by $H_*^{\mathcal{T}}(T)$ the homology objects of T (with values in \mathcal{B}) with respect to the t -structure of \mathcal{T} .

We give ourselves a (covariant) triangulated functor $R : \mathrm{DM}_{\mathrm{gm}}^{\mathrm{eff}} \otimes \mathbb{Q} \rightarrow \mathcal{T}$. (Note: this implies that K is of characteristic zero.)

Let $X \in \mathrm{Sm}$: for $i \in \mathbb{Z}$, we write $H_i^R(X) := H_i^{\mathcal{T}}(R(M(X)))$. We assume:

15.4.1. Hypothesis. *If X is smooth projective of dimension $d \leq 1$, we have*

- (1) $RM(X) \in \mathcal{T}_{[-2d, 0]}$ (in particular, $H_i^R(X) = 0$ for $i \notin [-2d, 0]$).
- (2) If $d = 1$ and E is the field of constants of X , the map $H_0^R(X) \rightarrow H_0^R(\mathrm{Spec} E)$ is an isomorphism.
- (3) If $d = 1$, E is the field of constants of X and $f : X \rightarrow \mathbb{P}_E^1$ is a nonconstant rational function, the map $f_* : H_2^R(X) \rightarrow H_2^R(\mathbb{P}_E^1)$ is an isomorphism.

15.4.2. Proposition. *Under Hypothesis 15.4.1, the composition*

$$\mathcal{M}_1 \otimes \mathbb{Q} \rightarrow D^b(\mathcal{M}_1) \otimes \mathbb{Q} \xrightarrow{\mathrm{Tot}} \mathrm{DM}_{\mathrm{gm}}^{\mathrm{eff}} \otimes \mathbb{Q} \xrightarrow{R} \mathcal{T}$$

has image in the heart \mathcal{B} of \mathcal{T} . The corresponding functor

$$R_1 : \mathcal{M}_1 \otimes \mathbb{Q} \rightarrow \mathcal{B}$$

is exact. If moreover

$$(W) \quad H_i^R(X) \in \mathcal{B}_{-i} \text{ for all } i \in \mathbb{N}$$

for X smooth projective with $\dim X \leq 1$, then R_1 respects the splittings of $\mathcal{M}_1 \otimes \mathbb{Q}$ and \mathcal{B} in the sense of Definition E.9.1 (2).

Proof. By definition of a t -structure, the first assertion will hold provided $R \circ \mathrm{Tot}(N) \in \mathcal{B}$ for any 1-motive N of pure weight. We are then left with lattices ($[L \rightarrow 0]$), tori ($[0 \rightarrow T]$) and abelian varieties ($[0 \rightarrow A]$). Moreover, since we work with rational coefficients, we may assume that $L = R_{E/k}\mathbb{Z}$ and $T = R_{E/k}\mathbb{G}_m$ for a finite separable extension E/k .

We have $\mathrm{Tot}([L \rightarrow 0]) = M(\mathrm{Spec} E)$ and $\mathrm{Tot}([0 \rightarrow T]) = M(\mathrm{Spec} E)(1)$; hence $R \mathrm{Tot}([L \rightarrow 0]) = RM(\mathrm{Spec} E) = H_0^R(\mathrm{Spec} E)[0] \in \mathcal{B}$ by Hypothesis 15.4.1 (1). On the other hand, since the only idempotents in $\mathrm{End}(M(\mathbb{P}_E^1))$ are the Künneth idempotents, yielding the decomposition $M(\mathbb{P}_E^1) = M(\mathrm{Spec} E) \oplus M(\mathrm{Spec} E)(1)[2]$, we get $R \mathrm{Tot}([0 \rightarrow T]) = H_2^R(\mathbb{P}_E^1) \in \mathcal{B}$, and also $H_1^R(\mathbb{P}_E^1) = 0$, by Hypothesis 15.4.1 (2).

For abelian varieties, we may reduce to the case of Jacobians of curves. Recall Voevodsky's functor [88, 2.1.4]

$$(15.3) \quad \Phi : \text{Chow}^{\text{eff}} \boxtimes \mathbb{Q} \rightarrow \text{DM}_{\text{gm}}^{\text{eff}} \boxtimes \mathbb{Q}$$

such that $\Phi(h(X)) = M(X)$ for any smooth projective X : this is a full embedding by the cancellation theorem of [87].

Let C be a smooth projective geometrically connected k -curve: the choice of a closed point $c \in C$ determines a Chow-Künneth decomposition. Also, $\text{Tot}([0 \rightarrow J(C)]) \simeq \Phi(h_1(C))[-1]$ as a Chow-Künneth direct summand of $M(C)[-1]$ (see 2.6.a). This already shows that $R\text{Tot}([0 \rightarrow J(C)]) \in \mathcal{T}_{[-1,1]}$, by Hypothesis 15.4.1 (1).

Choose a nonconstant rational function $f : C \rightarrow \mathbb{P}^1$, and let $x = f(c) \in \mathbb{P}^1$. If π_i^c and π_i^x are the corresponding Chow-Künneth projectors, we clearly have $f_*\pi_0^c = \pi_0^x f_*$. Hence the matrix of f_* on the decompositions $h(C) = h_0(C) \oplus h_1(C) \oplus h_2(C)$ and $h(\mathbb{P}^1) = h_0(\mathbb{P}^1) \oplus h_2(\mathbb{P}^1)$ is of the form

$$f_* = \begin{pmatrix} 1 & 0 & * \\ 0 & 0 & 1 \end{pmatrix}$$

if we identify $h_0(C), h_0(\mathbb{P}^1)$ with $\mathbf{1}$ and $h_2(C), h_2(\mathbb{P}^1)$ with \mathbb{L} . (One can compute that $*$ equals $\pm(f^{-1}(f(c)) - \deg f \cdot c)$, as an element of $J(C)(k) \otimes \mathbb{Q}$.) Thus we have an exact triangle in $\text{DM}_{\text{gm}}^{\text{eff}} \otimes \mathbb{Q}$:

$$\text{Tot}([0 \rightarrow J(C)][1]) \rightarrow M(C) \xrightarrow{f_*} M(\mathbb{P}^1) \xrightarrow{+1}$$

hence a long exact sequence in \mathcal{B} :

$$\dots \rightarrow H_{i+1}^R(\mathbb{P}^1) \rightarrow H_i^{\mathcal{T}}(\text{Tot}([0 \rightarrow J(C)][1])) \rightarrow H_i^R(C) \xrightarrow{f_*} H_i^R(\mathbb{P}^1) \rightarrow \dots$$

Using now the computation of $H_*^R(\mathbb{P}^1)$ and Hypothesis 15.4.1 (3), we find:

$$H_i^{\mathcal{T}}(\text{Tot}([0 \rightarrow J(C)][1])) = \begin{cases} 0 & i \neq 1 \\ H_1^R(C) & i = 1. \end{cases}$$

This shows that $R\text{Tot}([0 \rightarrow J(C)]) \in \mathcal{B}$, hence the functor $R_1 : \mathcal{M}_1 \otimes \mathbb{Q} \rightarrow \mathcal{B}$. By Lemma E.19.1, the composition

$$D^b(\mathcal{M}_1) \otimes \mathbb{Q} \xrightarrow{\text{Tot}} \text{DM}_{\text{gm}}^{\text{eff}} \otimes \mathbb{Q} \xrightarrow{R} \mathcal{T}$$

is t -exact relatively the canonical t -structure on $D^b(\mathcal{M}_1) \otimes \mathbb{Q}$ (the motivic one, with heart $\mathcal{M}_1 \otimes \mathbb{Q}$), and its restriction R_1 to the hearts is exact.

If Condition (W) is verified, the proof shows that R_1 respects the splittings of $\mathcal{M}_1 \otimes \mathbb{Q}$ and \mathcal{B} in the sense of Definition E.9.1 (2). \square

15.5. The base change theorem. Let $K, \mathcal{B}, \mathcal{T}, R$ be as in 15.4. If X is a smooth projective k -variety, we set as before

$$H_i^R(X) := H_i^{\mathcal{T}}(R(M(X)))$$

where $H_*^{\mathcal{T}}$ denotes the homology functors with values in \mathcal{B} defined by the t -structure on \mathcal{T} .

15.5.1. Remark. Let X be smooth projective and $x \in X$ be a closed point. Let $E = k(x)$ be the residue field of X . We have an Albanese map $a_x : X_E \rightarrow (\mathcal{A}_{X/k}^0)_E$, hence a map in $\mathrm{DM}_{\mathrm{gm}}^{\mathrm{eff}} \otimes \mathbb{Q}$:

$$\alpha_x : M(X) \xrightarrow{{}^t M(p)} M(X_E) \xrightarrow{M(a_x)} M((\mathcal{A}_{X/k}^0)_E) \xrightarrow{d^{-1}M(q)} M(\mathcal{A}_{X/k}^0)$$

where $d = [E : k]$, $p : X_E \rightarrow X$ and $q : (\mathcal{A}_{X/k}^0)_E \rightarrow \mathcal{A}_{X/k}^0$ are the projections and ${}^t M(p)$ is the transpose of the graph of (the finite surjective morphism) p . This map depends on the choice of x , but two choices differ by a translation on $\mathcal{A}_{X/k}^0$.

Thus we get maps

$$(\alpha_x)_* : H_i^R(X) \rightarrow H_i^R(\mathcal{A}_{X/k}^0).$$

If the (duals of the) H_i^R define a “classical” Weil cohomology theory in the sense of [1, 3.4], then $(\alpha_x)_*$ does not depend on the choice of x for $i > 0$. We won’t need this fact, but merely the fact that for two choices of x , the corresponding $(\alpha_x)_*$ differ by an isomorphism.

15.5.2. Hypothesis. We assume Hypothesis 15.4.1, and moreover:

- (1) The t -structure on \mathcal{T} is bounded.
- (2) The restriction of R_1 to $(\mathcal{M}_1 \otimes \mathbb{Q})_{-2}$ induces an equivalence of categories $(\mathcal{M}_1 \otimes K)_{-2} \xrightarrow{\sim} \mathcal{B}_{\mathbb{L}}$.
- (3) For any X smooth projective:
 - (i) $H_i^R(X) := H_i(R(M(X))) = 0$ for $i < 0$.
 - (ii) $H_i^R(X) \in \mathcal{B}_{-i}$ for $i \geq 0$.
 - (iii) $H_0^R(X) \xrightarrow{\sim} H_0^R(\pi_0(X))$.
 - (iv) The choice of a closed point of X induces an isomorphism $(\alpha_x)_* : H_1^R(X) \xrightarrow{\sim} H_1^R(\mathcal{A}_{X/k}^0)$. (This does not depend on the choice of x : see Remark 15.5.1.)

Let $R_1 := R \mathrm{Tot}$. Using Proposition 15.3.4 and Section 15.1, we get from this equality a base change morphism

$$(15.4) \quad v : \mathrm{LAlb}^{\mathcal{T}} R \Rightarrow R_1 \mathrm{LAlb}^{\mathbb{Q}}.$$

15.5.3. Lemma. *Under 15.5.2, we have a commutative diagram*

$$\begin{array}{ccc} \mathrm{Hom}_{\mathrm{DM}_{\mathrm{gm}}^{\mathrm{eff}} \otimes \mathbb{Q}}(M(X), \mathbb{Z}(1)[2]) & \xrightarrow{R} & \mathrm{Hom}_{\mathcal{T}}(R(X), R(\mathbb{Z}(1))[2]) \\ \alpha \downarrow & & \beta \downarrow \\ \mathrm{Hom}_{\mathcal{B}}(R_1 L_2 \mathrm{Alb}^{\mathbb{Q}}(X), \Lambda) & \xrightarrow{v^*} & \mathrm{Hom}_{\mathcal{B}}(\mathrm{Alb}^{\mathcal{T}} H_2^R(X), \Lambda) \end{array}$$

for any smooth projective variety X , where $\Lambda := R_1([0 \rightarrow \mathbb{G}_m])$.

Proof. Playing with the adjunctions and t -structures, we have chains of maps

$$\begin{aligned} (15.5) \quad & (\mathrm{DM}_{\mathrm{gm}}^{\mathrm{eff}} \otimes \mathbb{Q})(M(X), \mathbb{Z}(1)[2]) \simeq (D^b(\mathcal{M}_1) \otimes \mathbb{Q})(L\mathrm{Alb}^{\mathbb{Q}}(X), [0 \rightarrow \mathbb{G}_m][2]) \\ & \xrightarrow{^t H_2} (\mathcal{M}_1 \otimes \mathbb{Q})(L_2 \mathrm{Alb}^{\mathbb{Q}}(X), [0 \rightarrow \mathbb{G}_m]) \xrightarrow{R_1} \mathcal{B}(R_1 L_2 \mathrm{Alb}^{\mathbb{Q}}(X), \Lambda) \end{aligned}$$

and

$$(15.6) \quad \mathcal{T}(R(X), R(\mathbb{Z}(1))[2]) \simeq \mathcal{T}(R(X), \Lambda[2]) \xrightarrow{H_2^R} \mathcal{B}(H_2^R(X), \Lambda) \simeq \mathcal{B}(\mathrm{Alb}^{\mathcal{T}} H_2^R(X), \Lambda).$$

This defines respectively α and β . By following the various adjunction isomorphisms, the diagram is commutative as claimed. \square

15.5.4. Remark. Note that $\mathrm{Hom}_{\mathcal{B}}(R_1 L_2 \mathrm{Alb}^{\mathbb{Q}}(X), \Lambda) = \mathrm{NS}(\overline{X})^{G_E} \otimes K$ by Corollary 10.2.3 and Hypothesis 15.5.2 (2). Also, we have an injection $\mathrm{Hom}_{\mathcal{B}}(\mathrm{Alb}^{\mathcal{T}} H_2^R(X), \Lambda) \hookrightarrow \mathrm{Hom}_{\mathcal{B}}(H_2^R(X), \Lambda)$ by (the proof of) Proposition 15.3.4 (2). If \mathcal{B} sits in a larger “non-effective” category \mathcal{B}' which carries a duality, the latter group may be interpreted as $\mathrm{Hom}_{\mathcal{B}}(K_{\mathcal{B}'}, H_R^2(X)(1))$, with $K_{\mathcal{B}} := R_1(\mathbb{Z})$. Finally, the composition βR in the diagram of Lemma 15.5.3, followed by the latter inclusion, is easily checked to be the cycle class map in the classical cases. So the bottom row of this diagram contains an abstract argument that algebraic equivalence is weaker than homological equivalence in codimension 1.

On the other hand, it is known that algebraic and numerical equivalences coincide rationally in codimension 1. Thus, if H_*^R defines an adequate equivalence relation on algebraic cycles, we automatically get that v^* is injective in the diagram of Lemma 15.5.3. This will be the case if H_R^* defines a Weil cohomology theory, which will follow if \mathcal{B} and \mathcal{T} can be extended to categories satisfying natural extra axioms (tensor structure, duality), cf. Cisinski-Dégliše [22].

15.5.5. Theorem. *Under 15.5.2,*

- (1) *The base change morphism v is an isomorphism in weights 0 and -1 .*
- (2) *Let $(\mathrm{DM}_{\mathrm{gm}}^{\mathrm{eff}})'$ be the thick subcategory of $\mathrm{DM}_{\mathrm{gm}}^{\mathrm{eff}} \boxtimes \mathbb{Q}$ generated by the $M(X)$ where X is a smooth projective variety such that, for any finite extension E/k , the map v^* of Lemma 15.5.3 is injective for X_E (see Remark 15.5.4) and the “geometric cycle class map”*

$$(15.7) \quad \mathrm{Pic}(X_E) \otimes K = \mathrm{Hom}_{\mathrm{DM}_{\mathrm{gm}}^{\mathrm{eff}} \otimes \mathbb{Q}}(M(X_E), \mathbb{Z}(1)[2]) \otimes K$$

$$\xrightarrow{R} \mathrm{Hom}_{\mathcal{T}}(R(X_E), R(\mathbb{Z}(1))[2]) \xrightarrow{H_2^R} \mathrm{Hom}_{\mathcal{B}}(H_2^R(X_E), R(\mathbb{Z}(1)))$$

is surjective. Then the restriction of v to $(\mathrm{DM}_{\mathrm{gm}}^{\mathrm{eff}})'$ is an isomorphism.

Proof. (1) By de Jong’s theorem, it suffices to prove the statement for $M = M(X)$, X smooth projective. Thus we have to prove that, for all $i \in \mathbb{Z}$, the map

$$\mathrm{Alb}^{\mathcal{T}} H_i^R(X) \simeq H_i^{\mathcal{T}}(\mathrm{LAlb}^{\mathcal{T}} R(X)) \rightarrow R_1 \mathrm{L}_i \mathrm{Alb}^{\mathbb{Q}}(X)$$

defined by $v_{M(X)}$ is an isomorphism in weights 0 and -1 . Here we used the t -exactness of $\mathrm{LAlb}^{\mathcal{T}}$, cf. Proposition 15.3.4 (2).

For $i < 0$ (resp. $i > 2$), both sides are 0 by Corollary 10.2.3 and 15.5.2 (3) (i) (resp. (ii)). For $i = 0, 1$, (iii) and (iv) imply that the map is an isomorphism. Finally, 15.5.2 (3) (ii) implies that $H_2^R(X)$ is pure of weight -2 and so is $\mathrm{L}_2 \mathrm{Alb}^{\mathbb{Q}}(X)$, hence the statement is still true in this case.

(2) It is sufficient to show that, for any smooth projective X verifying the condition of (2), the morphism $\mathrm{Alb}^{\mathcal{T}} H_2^R(X) \rightarrow R_1 \mathrm{L}_2 \mathrm{Alb}^{\mathbb{Q}}(X)$ is an isomorphism. Note that both sides are Lefschetz objects: hence, by Yoneda’s lemma, it suffices to show that for any Lefschetz object Θ , the map

$$(15.8) \quad \mathrm{Hom}(R_1 \mathrm{L}_2 \mathrm{Alb}^{\mathbb{Q}}(X), \Theta) \rightarrow \mathrm{Hom}(\mathrm{Alb}^{\mathcal{T}} H_2^R(X), \Theta)$$

is an isomorphism.

Without loss of generality, we may assume that $\Theta = R_1([0 \rightarrow T])$ with $T = R_{E/k} \mathbb{G}_m = M(\mathrm{Spec} E) \otimes \mathbb{Z}(1)[1]$. In the commutative diagram of Lemma 15.5.3 for X_E , the composition of β and R is surjective by the surjectivity of (15.7). Therefore (15.8) is surjective. By assumption, (15.8) is also injective. This concludes the proof. \square

16. COMPARING REALISATIONS

Let $K, \mathcal{T}, \mathcal{B}$ be as in 15.3 and R be as in 15.4. We now give ourselves

- a t -exact functor $D^b(\mathcal{B}) \rightarrow \mathcal{T}$ which is the identity on \mathcal{B}
- an exact functor $\underline{\mathbb{T}} : \mathcal{M}_1 \otimes \mathbb{Q} \rightarrow \mathcal{B}$.

We suppose that $\underline{\mathbb{T}}$ respects the splittings of $\mathcal{M}_1 \otimes \mathbb{Q}$ and \mathcal{B} in the sense of Definition E.9.1 (see Proposition 15.2.1 for the weight filtration on $\mathcal{M}_1 \otimes \mathbb{Q}$). We still write $\underline{\mathbb{T}}$ for its trivial extension

$$D^b(\mathcal{M}_1) \otimes \mathbb{Q} \xrightarrow{\underline{\mathbb{T}}} D^b(\mathcal{B}) \rightarrow \mathcal{T}.$$

We would like to compare $\underline{\mathbb{T}}$ with $R_1 = R \text{ Tot}$. The aim of this section is to discuss the existence and uniqueness of a natural isomorphism between these two functors.

16.1. A full embedding. Let Chow be the category of Chow motives over k , $\text{Chow} \boxtimes \mathbb{Q}$ the category of Chow motives with rational coefficients and $\text{Chow}^{\text{eff}} \boxtimes \mathbb{Q}$ the full subcategory of effective Chow motives. We take the covariant convention for composition of correspondences: the functor

$$X \mapsto h(X)$$

from smooth projective varieties to $\text{Chow}^{\text{eff}} \boxtimes \mathbb{Q}$ is covariant.

16.1.1. Lemma. *Let $d_{\leq 1} \text{Chow}^{\text{eff}} \boxtimes \mathbb{Q}$ be the thick subcategory of $\text{Chow}^{\text{eff}} \boxtimes \mathbb{Q}$ generated by motives of curves. Then*

- (1) $d_{\leq 1} \text{Chow}^{\text{eff}} \boxtimes \mathbb{Q}$ consists of those objects that may be written as a direct sum of Chow motives of the following type:
 - an Artin motive;
 - a motive of the form $h_1(A)$ for A an abelian variety;
 - a motive of the form $M \otimes \mathbb{L}$, where M is an Artin motive and \mathbb{L} is the Lefschetz motive.
- (2) The composition

$$d_{\leq 1} \text{Chow}^{\text{eff}} \boxtimes \mathbb{Q} \xrightarrow{\iota} \text{Chow}^{\text{eff}} \boxtimes \mathbb{Q} \xrightarrow{\Phi} \text{DM}_{\text{gm}}^{\text{eff}} \boxtimes \mathbb{Q} \xrightarrow{\text{LAlb}^{\mathbb{Q}}} D^b(\mathcal{M}_1 \otimes \mathbb{Q})$$

is fully faithful, where Φ is as in (15.3). It yields a naturally commutative diagram of full embeddings:

$$\begin{array}{ccc} d_{\leq 1} \text{Chow}^{\text{eff}} \boxtimes \mathbb{Q} & \xrightarrow{\Phi_{(1)}} & D^b(\mathcal{M}_1 \otimes \mathbb{Q}) \\ \downarrow \iota & & \downarrow \text{Tot}^{\mathbb{Q}} \\ \text{Chow}^{\text{eff}} \boxtimes \mathbb{Q} & \xrightarrow{\Phi} & \text{DM}_{\text{gm}}^{\text{eff}} \boxtimes \mathbb{Q} \end{array}$$

with $\Phi_{(1)} = \text{LAlb}^{\mathbb{Q}} \circ \Phi \circ \iota$.

Proof. (1) is clear by the Chow-Künneth decomposition for motives of curves and the fact that any abelian variety is a direct summand of the Jacobian of a curve, up to isogeny.

(2) We have a naturally commutative diagram

$$\begin{array}{ccccc}
d_{\leq 1} \text{Chow}^{\text{eff}} \boxtimes \mathbb{Q} & \xrightarrow{\Phi} & d_{\leq 1} \text{DM}_{\text{gm}}^{\text{eff}} \boxtimes \mathbb{Q} & & \\
\downarrow & & \downarrow & & \\
\text{Chow}^{\text{eff}} \boxtimes \mathbb{Q} & \xrightarrow{\Phi} & \text{DM}_{\text{gm}}^{\text{eff}} \boxtimes \mathbb{Q} & \xrightarrow{\text{LAlb}} & D^b(\mathcal{M}_1 \otimes \mathbb{Q})
\end{array}$$

where the vertical functors are full embeddings. The lemma now boils down to the fact that the restriction of $\text{LAlb}^{\mathbb{Q}}$ to $d_{\leq 1} \text{DM}_{\text{gm}}^{\text{eff}} \boxtimes \mathbb{Q}$ is quasi-inverse to Tot (Corollary 6.2.2). \square

16.1.2. Remark. Of course, we can now define the full embedding $\Phi_{(1)}$ of Lemma 16.1.1 without reference to $\text{DM}_{\text{gm}}^{\text{eff}}$ or $\text{LAlb}^{\mathbb{Q}}$: this functor sends

- (1) An Artin motive M to the 1-motive $[L \rightarrow 0]$, where L is the permutation Galois-module associated to M .
- (2) If A is an abelian variety, $h_1(A)$ to $[0 \rightarrow A][1]$.
- (3) A Lefschetz motive $M \otimes \mathbb{L}$ to $[0 \rightarrow L \otimes \mathbb{G}_m][2]$, where L is as in (1).

16.2. Extending a natural isomorphism.

16.2.1. Hypothesis. We assume given, for every smooth projective curve C , an isomorphism $v_C : R(M(C)) \xrightarrow{\sim} \underline{T}(\Phi_{(1)}h(C))$ with the following two properties:

- (i) If C' is another such curve, then for any Chow correspondence $\alpha \in CH^1(C \times C') \otimes \mathbb{Q}$, the diagram

$$\begin{array}{ccc}
R(M(C)) & \xrightarrow{\alpha_*} & R(M(C')) \\
v_C \downarrow & & v_{C'} \downarrow \\
\underline{T}(\Phi_{(1)}h(C)) & \xrightarrow{\alpha_*} & \underline{T}(\Phi_{(1)}h(C'))
\end{array}
\tag{16.1}$$

commutes.

Let E/k be a finite extension, and let $h(E) := h(\text{Spec } E)$, $\mathbb{Z}_E := M(\text{Spec } E) = \Phi h(E)$, $[\mathbb{Z}_E \rightarrow 0] := [\mathbb{Z}[G_k/G_E] \rightarrow 0] = \text{LAlb}(\mathbb{Z}_E)$. For $C = \mathbb{P}_E^1$ with $M(\mathbb{P}_E^1) = \mathbb{Z}_E \oplus \mathbb{Z}_E(1)[2]$, $v_{\mathbb{P}_E^1}$ and the commutation of (16.1) for Chow correspondences from \mathbb{P}_E^1 to itself yield two isomorphisms $v_{h(E)} : R(\mathbb{Z}_E) \xrightarrow{\sim} \underline{T}([Z_E \rightarrow 0])$ and $v_{\mathbb{L} \otimes h(E)} : R(\mathbb{Z}_E(1)[2]) \xrightarrow{\sim} \underline{T}([0 \rightarrow \mathbb{Z}_E \otimes \mathbb{G}_m][1])$. The second property we require is:

- (ii) For any $E/k, E'/k$ and any $\beta \in \text{Hom}_{\text{DM}}(\mathbb{Z}_E, \mathbb{Z}_{E'}(1)[1]) = (E \otimes_k E')^* \otimes \mathbb{Q}$, the diagram

$$(16.2) \quad \begin{array}{ccc} R(\mathbb{Z}_E) & \xrightarrow{\beta_*} & R(\mathbb{Z}_{E'}(1)[1]) \\ v_{h(E)} \downarrow & & v_{\mathbb{L}}[-1] \downarrow \\ \underline{\mathbb{T}}([\mathbb{Z}_E \rightarrow 0]) & \xrightarrow{\beta_*} & \underline{\mathbb{T}}([0 \rightarrow \mathbb{Z}_{E'} \otimes \mathbb{G}_m]) \end{array}$$

commutes.

The aim of this subsection is to prove the following

16.2.2. Theorem. *Let as before $R_1 = R \text{ Tot}$. Under Hypotheses 15.4.1 and 16.2.1, there exists at most one natural isomorphism $u : R_1 \Rightarrow \underline{\mathbb{T}}$ such that, for every smooth projective curve C , $v_C = u_{\Phi_{(1)}h(C)}$. This isomorphism exists if and only if a certain obstruction (see (5) in the proof below) vanishes.*

Proof. Recall first that, under Hypothesis 15.4.1, $R_1(\mathcal{M}_1 \otimes \mathbb{Q}) \subseteq \mathcal{B}$ (Proposition 15.4.2). If $u : R_1 \Rightarrow \underline{\mathbb{T}}$ is a natural transformation, then it is determined by its restriction u_1 to $\mathcal{M}_1 \otimes \mathbb{Q}$ by Lemma E.22.1; moreover, u is an isomorphism if and only if u_1 is (*ibid.*). Therefore, we only need to discuss the existence and uniqueness of u_1 .

The commutativity of the diagrams (16.1) means that v defines a natural transformation on the category of smooth projective curves provided with Chow correspondences. We proceed in several steps:

- (1) Extend v to $d_{\leq 1} \text{Chow}^{\text{eff}} \boxtimes \mathbb{Q}$, yielding natural isomorphisms

$$v_h : R\Phi(h) \xrightarrow{\sim} \underline{\mathbb{T}}(\Phi_{(1)}(h)), \quad h \in d_{\leq 1} \text{Chow}^{\text{eff}} \boxtimes \mathbb{Q}$$

and diagrams like (16.2) with \mathbb{Z}_E (*resp.* $\mathbb{Z}_{E'}(1)[1]$) replaced by $\Phi(h)$ (*resp.* by $\Phi(h' \otimes \mathbb{L})[-1]$ for any Artin motives h, h' . Here \mathbb{L} is the Lefschetz motive.

- (2) Define u on 1-motives of pure weight.
 (3) Extend u to 1-motives of weights $\in \{-2, -1\}$ and $\in \{-1, 0\}$.
 (4) Extend u to 1-motives of weights $\in \{-2, 0\}$.
 (5) Get the obstruction to extend u to all 1-motives.

(1) is formal.

(2) Let N be of pure weight $-w$. Then $N = \text{LAlb } \Phi(h)[-w]$ for a unique $h \in d_{\leq 1} \text{Chow}^{\text{eff}} \boxtimes \mathbb{Q}$ of weight $-w$. Conversely, we have $\Phi(h) = \text{Tot } N[w]$. Define $u_N = v_h[w]$: this indeed defines a natural transformation thanks to Lemma 16.1.1 (2).

(3) Start with the case of weights $\in \{-2, -1\}$. By Theorem E.10.2 (2), we have to show that given a 1-motive $[0 \rightarrow G]$, with G extension

of A by T , the diagram

$$\begin{array}{ccc} \underline{\mathbb{T}}'([0 \rightarrow A]) & \xrightarrow{[\underline{\mathbb{T}}'([0 \rightarrow G])]} & \underline{\mathbb{T}}'([0 \rightarrow T])[1] \\ u_{[0 \rightarrow A]} \downarrow & & u_{[0 \rightarrow T]}[1] \downarrow \\ \underline{\mathbb{T}}([0 \rightarrow A]) & \xrightarrow{[\underline{\mathbb{T}}([0 \rightarrow G])]} & \underline{\mathbb{T}}([0 \rightarrow T])[1] \end{array}$$

commutes.

We may write $[0 \rightarrow A] = \Phi_{(1)}(\mathbf{A})[-1]$ and $[0 \rightarrow T] = \Phi_{(1)}(\mathbf{T})[-2]$, where $\mathbf{A} = h_1(A)$ and $\mathbf{T} = \mathbb{L} \otimes \text{Hom}(\mathbb{G}_m, T)$ (\mathbb{L} the Lefschetz motive). Then the map

$$[0 \rightarrow A] \rightarrow [0 \rightarrow T][1]$$

corresponding to the extension $[0 \rightarrow G]$ comes from a map $\mathbf{A} \rightarrow \mathbf{T}$ by Lemma 16.1.1. By (1), the diagram

$$\begin{array}{ccc} R(\Phi \mathbf{A}) & \longrightarrow & R(\Phi \mathbf{T}) \\ v_{\Phi \mathbf{A}} \downarrow & & v_{\Phi \mathbf{T}} \downarrow \\ \underline{\mathbb{T}}(\Phi_{(1)} \mathbf{A}) & \longrightarrow & \underline{\mathbb{T}}(\Phi_{(1)} \mathbf{T}) \end{array}$$

commutes. This is equivalent to the fact we request.

In the case of weights $\in \{-1, 0\}$, given a 1-motive $[L \xrightarrow{f} A]$ we have similarly to check that the diagram

$$\begin{array}{ccc} R(\Phi(L \otimes \mathbf{1})) & \xrightarrow{f_*} & R(\Phi \mathbf{A}) \\ v_{\Phi(L \otimes \mathbf{1})} \downarrow & & v_{\Phi \mathbf{A}} \downarrow \\ \underline{\mathbb{T}}(\Phi_{(1)}(L \otimes \mathbf{1})) & \xrightarrow{f_*} & \underline{\mathbb{T}}(\Phi_{(1)} \mathbf{A}) \end{array}$$

commutes, which also follows from (1).

(4) We have to prove that, given a 1-motive $N = [L \xrightarrow{f} T]$ with T a torus, the diagram

$$\begin{array}{ccc} \underline{\mathbb{T}}'([0 \rightarrow T]) & \xrightarrow{[\underline{\mathbb{T}}'(N)]} & \underline{\mathbb{T}}'([L \rightarrow 0])[1] \\ u_{[0 \rightarrow T]} \downarrow & & u_{[L \rightarrow 0]}[1] \downarrow \\ \underline{\mathbb{T}}([0 \rightarrow T]) & \xrightarrow{[\underline{\mathbb{T}}(N)]} & \underline{\mathbb{T}}([L \rightarrow 0])[1] \end{array}$$

commutes. As in (5), this is a translation of (1).

(5) By Theorem E.12.1 (3), we now get an obstruction to glueing of the following form: for any 1-motive $[L \rightarrow G]$, a bilinear map

$$\begin{aligned} \text{Ext}_{\mathcal{M}_1 \otimes \mathbb{Q}}^1([L \rightarrow 0], [0 \rightarrow A]) \times \text{Ext}_{\mathcal{M}_1 \otimes \mathbb{Q}}^1([0 \rightarrow A], [0 \rightarrow T]) \\ \rightarrow \text{Ext}_{\mathcal{B}}^1(\underline{\mathbb{T}}'([L \rightarrow 0]), \underline{\mathbb{T}}'([0 \rightarrow T])). \end{aligned}$$

In case $L = \mathbb{Z}$ and $T = \mathbb{G}_m$, we may rewrite this as

$$A(k) \otimes \mathbb{Q} \times A'(k) \otimes \mathbb{Q} \rightarrow \mathrm{Ext}_{\mathcal{B}}^1(R\mathbb{Z}, R\mathbb{Z}(1)).$$

□

16.2.3. Remark. If we assume that the natural map

$$\mathrm{DM}_{\mathrm{gm}}^{\mathrm{eff}} \otimes \mathbb{Q}(\mathbb{Z}, \mathbb{Z}(1)[1]) \otimes K \xrightarrow{R} \mathcal{T}(R\mathbb{Z}, R\mathbb{Z}(1)[1]) = \mathrm{Ext}_{\mathcal{B}}^1(R\mathbb{Z}, R\mathbb{Z}(1))$$

is bijective, then the pairing in part (5) of the proof of Theorem 16.2.2 may be reformulated as a K -pairing

$$(16.3) \quad A(k) \otimes K \times A'(k) \otimes K \rightarrow k^* \otimes K.$$

Suppose that we have the above picture functorially over a base S , where S runs through smooth k -schemes essentially of finite type. By Proposition C.12.3, the category of smooth 1-motives $\mathcal{M}_1(\mathcal{O}_{S,s}^{sh}) \otimes \mathbb{Q}$ [25, 10.1.10] is of cohomological dimension ≤ 1 for any point $s \in S$. Hence the pairing (16.3) extends to a pairing of étale heaves

$$A \otimes K \times A' \otimes K \rightarrow \mathbb{G}_m \otimes K.$$

by Theorem E.12.1 (3). So we get in particular a homomorphism

$$\varphi_S : A(S) \otimes K \rightarrow \mathrm{Hom}(A'(k) \otimes K, \Gamma(S, \mathbb{G}_m) \otimes K)$$

natural in S .

For $S = A$, the right hand side is reduced to $\mathrm{Hom}(A'(k) \otimes K, k^* \otimes K)$, hence Yoneda's lemma implies that φ_S is constant, and therefore 0.

17. THE HODGE REALISATION

Let MHS denote the abelian category of (graded polarizable, \mathbb{Q} -linear) mixed Hodge structures. We take for \mathcal{B} the full subcategory $\mathrm{MHS}^{\mathrm{eff}}$ of MHS consisting of objects whose Hodge numbers are of type (i, j) with $i \leq 0$ and $j \leq 0$. For \mathcal{T} we take $D^b(\mathcal{B})$. The weight filtration provides \mathcal{B} with a weight filtration in the sense of Definition E.7.2. We take for $\mathcal{B}_{\mathbb{L}}$ the Hodge structures purely of type $(-1, -1)$. With the notation of §15.3, $\mathcal{B}_{(1)} = \mathrm{MHS}_{(1)}$ is the full subcategory of $\mathrm{MHS}^{\mathrm{eff}}$ given by mixed Hodge structures with possibly non zero Hodge numbers in the set $\{(0, 0), (0, -1), (-1, 0), (-1, -1)\}$.

17.1. LAlb $^{\mathcal{T}}$ for mixed Hodge structures. Note that MHS_{-2} is semi-simple since pure polarizable Hodge structures are. As a special case of Proposition 15.3.4, we therefore have:

17.1.1. Proposition. *The full embeddings $\iota : \mathrm{MHS}_{(1)} \hookrightarrow \mathrm{MHS}^{\mathrm{eff}}$ and $\iota : D^b(\mathrm{MHS}_{(1)}) \hookrightarrow D^b(\mathrm{MHS}^{\mathrm{eff}})$ have left adjoints $\mathrm{Alb}^{\mathrm{MHS}}$ and $\mathrm{LAlb}^{\mathrm{MHS}}$.*

□

17.1.2. Notation. For $H \in \text{MHS}^{\text{eff}}$ we shall write $H_{\leq 1}$ for $\text{Alb}^{\text{MHS}}(H)$. If H has Hodge numbers ≥ 0 then the dual $\underline{\text{Hom}}(H, \mathbb{Q}) \in \text{MHS}^{\text{eff}}$ is effective and we denote

$$H^{\leq 1} := \underline{\text{Hom}}(\underline{\text{Hom}}(H, \mathbb{Q})_{\leq 1}, \mathbb{Q}(1))$$

Denoting Pic^{MHS} the Cartier dual of Alb^{MHS} the latter $H^{\leq 1}$ translates in $\text{Pic}^{\text{MHS}}(\underline{\text{Hom}}(H, \mathbb{Q}))$.

17.1.3. Remark. In Deligne's notation [25, 10.4.1], for $H \in \text{MHS}$ we have:

- $\Pi_n(H)_{\mathbb{Q}} = H(n)_{\leq 1}$ if the Hodge numbers of H are $\leq n$
- $\text{I}(H)_{\mathbb{Q}} = H^{\leq 1}$ if the Hodge numbers of H are ≥ 0 .

Note that $H \mapsto H^{\leq 1}$ is actually right adjoint to the (full embedding) functor $H \mapsto H(-1)$ from $\text{MHS}_{(1)}$ to the full subcategory MHS_{eff} of MHS consisting of objects whose Hodge numbers are ≥ 0 .

17.2. Huber's Hodge realisation functor. We have A. Huber's realisation functor [42]

$$R_{\text{Hodge}} : \text{DM}_{\text{gm}}^{\text{eff}}(\mathbb{C}) \otimes \mathbb{Q} \rightarrow D^b(\text{MHS}).$$

This functor is contravariant. To get a covariant functor we compose it with the (exact) duality of $D^b(\text{MHS})$ sending H to $\underline{\text{Hom}}(H, \mathbb{Q})$. For X a smooth variety, we have $S_{\text{Hodge}}(M(X)) = R\Gamma(X, \mathbb{Q})$, and in particular its Hodge numbers are ≥ 0 . Thus we get a functor

$$R^{\text{Hodge}} : \text{DM}_{\text{gm}}^{\text{eff}}(\mathbb{C}) \otimes \mathbb{Q} \rightarrow D^b(\text{MHS}^{\text{eff}}).$$

Since $R_{\text{Hodge}}\Phi : \text{Chow}^{\text{eff}} \otimes \mathbb{Q} \rightarrow D^b(\text{MHS})$ is by construction isomorphic to the functor $X \mapsto R\Gamma(X, \mathbb{Q})$, the conditions of Hypothesis 15.4.1 are verified, as well as Condition (W) of Proposition 15.4.2. This proposition then shows that $R^{\text{Hodge}} \text{Tot}$ defines an exact functor

$$R_1^{\text{Hodge}} : \mathcal{M}_1(\mathbb{C}) \otimes \mathbb{Q} \rightarrow \text{MHS}_{(1)}.$$

Although this is irrelevant for our purpose, it is nice to know:

17.2.1. Theorem. *The functor R_1^{Hodge} is an equivalence of categories.*

Proof. Note that R_1^{Hodge} respects the splittings of $\mathcal{M}_1 \otimes \mathbb{Q}$ and $\text{MHS}_{(1)}$, by Proposition 15.4.2. Therefore we are in a position to apply Theorem E.14.1.

Let $W_i \mathcal{M}_1(\mathbb{C}) \otimes \mathbb{Q}$ and $W_i(\text{MHS}_{(1)})$ denote the full subcategories of objects pure of weight i ($i = 0, -1, -2$). We first check that R_1 induces equivalences of categories $W_i \mathcal{M}_1(\mathbb{C}) \otimes \mathbb{Q} \xrightarrow{\sim} W_i(\text{MHS}_{(1)})$. Note that these three categories are semi-simple. The cases $i = 0$ and $i = -2$ are obvious. For $i = -1$, it is known (*e.g.* from Deligne's equivalence

of categories [25, 10.1.3]) that any $H \in W_{-1}(\text{MHS}_{(1)})$ is a direct summand of some $H_1(C)$: this proves essential surjectivity. For the full faithfulness, we reduce to proving that, given two connected smooth projective curves, the map

$$\text{Hom}(J(C), J(C')) \rightarrow \text{Hom}(H_1(C), H_1(C'))$$

given by R_1^{Hodge} is the usual action of divisorial correspondences, which follows from the construction of R^{Hodge} .

(Note that these arguments provide natural isomorphisms of the restrictions of R_1^{Hodge} to $W_i\mathcal{M}_1(\mathbb{C}) \otimes \mathbb{Q}$ with Deligne's realisation functor.)

We are left to check the conditions of Theorem E.14.1 on isomorphisms of Hom and Ext groups. Note that the condition on Hom groups in (2) is empty because they are identically 0. For the Ext groups, since $\text{Ext}_{\text{MHS}}^2$ is identically 0, we reduce by (4) to prove that, for $N_m, N_n \in \mathcal{M}_1 \otimes \mathbb{Q}$ of pure weights $m < n$, the map

$$\text{Ext}_{\mathcal{M}_1 \otimes \mathbb{Q}}^1(N_n, N_m) \rightarrow \text{Ext}_{\text{MHS}}^1(R_1^{\text{Hodge}}(N_n), R_1^{\text{Hodge}}(N_m))$$

is bijective.

Since $\text{Tot} : D^b(\mathcal{M}_1 \otimes \mathbb{Q}) \rightarrow \text{DM}_{\text{gm}}^{\text{eff}} \boxtimes \mathbb{Q}$ is fully faithful, it suffices to prove that the map

$$\begin{aligned} \text{Hom}_{\text{DM}_{\text{gm}}^{\text{eff}} \boxtimes \mathbb{Q}}(\text{Tot}(N_n), \text{Tot}(N_m)[1]) \\ \rightarrow \text{Hom}_{D^b(\text{MHS})}(R^{\text{Hodge}} \text{Tot}(N_n), R^{\text{Hodge}} \text{Tot}(N_m)[1]) \end{aligned}$$

is bijective. We distinguish 3 cases:

- (i) $(m, n) = (-1, 0)$. Without loss of generality, we may assume $N_0 = [\mathbb{Z} \rightarrow 0]$, $N_{-1} = [0 \rightarrow J_C]$ for a smooth projective curve C . Then $\text{Tot}(N_{-1}) = \Phi h_1(C)[-1]$, $R^{\text{Hodge}} \text{Tot}(N_{-1}) = H_1(C)$ and we are looking at the map

$$\text{Hom}_{\text{DM} \boxtimes \mathbb{Q}}(\mathbb{Z}, \Phi h_1(C)) \rightarrow \text{Hom}_{D^b(\text{MHS})}(\mathbb{Z}, H_1(C)[1]).$$

By Poincaré duality, this map is equivalent to the map

$$\begin{aligned} J_C(\mathbb{C}) \otimes \mathbb{Q} = \text{Hom}_{\text{DM} \boxtimes \mathbb{Q}}(\Phi h^1(C), \mathbb{Z}(1)[2]) \\ \rightarrow \text{Hom}_{D^b(\text{MHS})}(H^1(C), \mathbb{Z}(1)[1]) = \text{Ext}_{\text{MHS}}^1(H^1(C), \mathbb{Z}(1)) \end{aligned}$$

which coincides with the Abel-Jacobi map¹⁴, hence is bijective.

- (ii) $(m, n) = (-2, -1)$. Without loss of generality, we may assume $N_{-2} = [0 \rightarrow \mathbb{G}_m]$, $N_{-1} = [0 \rightarrow J_C]$ for a smooth projective

¹⁴because, by construction, the restriction of Huber's functor to pure motives is the "usual" Hodge realisation functor.

curve C . Then $\mathrm{Tot}(N_{-1}) = \Phi h^1(C)[-1]$, $R\mathrm{Tot}(N_{-1}) = H^1(C)$ and we are looking at the map

$$\mathrm{Hom}_{\mathrm{DM} \boxtimes \mathbb{Q}}(\Phi h^1(C)\mathbb{Z}(1)[2]) \rightarrow \mathrm{Hom}_{D^b(\mathrm{MHS})}(H^1(C), \mathbb{Z}(1)[1])$$

which is the Abel-Jacobi map as in (ii).

- (iii) $(m, n) = (-2, 0)$. Without loss of generality, we may assume $N_{-2} = [0 \rightarrow \mathbb{G}_m]$, $N_0 = [\mathbb{Z} \rightarrow 0]$. We are now looking at the map

$$\begin{aligned} \mathbb{C}^* \otimes \mathbb{Q} &= \mathrm{Hom}_{\mathrm{DM} \boxtimes \mathbb{Q}}(\mathbb{Z}, \mathbb{Z}(1)[1]) \\ &\rightarrow \mathrm{Hom}_{D^b(\mathrm{MHS})}(\mathbb{Z}, \mathbb{Z}(1)[1]) = \mathrm{Ext}_{\mathrm{MHS}}^1(\mathbb{Z}, \mathbb{Z}(1)) \end{aligned}$$

which is again the usual isomorphism, by definition of Huber's realisation functor.

□

17.3. Deligne's conjecture. From Theorem 17.2.1 and (15.4) we get, for any $M \in \mathrm{DM}_{\mathrm{gm}}^{\mathrm{eff}}(\mathbb{C}) \otimes \mathbb{Q}$, a natural map

$$(17.1) \quad R^{\mathrm{Hodge}}(M)_{\leq 1} \rightarrow R_1^{\mathrm{Hodge}} \mathrm{LAlb}(M).$$

Taking homology of both sides, we get comparison maps

$$(17.2) \quad H_i(R^{\mathrm{Hodge}}(M))_{\leq 1} \rightarrow R_1^{\mathrm{Hodge}} \mathrm{L}_i \mathrm{Alb}(M)$$

for all $i \in \mathbb{Z}$. From Theorem 15.5.5 and the Lefschetz 1-1 theorem, we now get:

17.3.1. Theorem. *The maps (17.1) and (17.2) are isomorphisms for all motives M .* □

This theorem recovers the results stated in [7] with rational coefficients.

The isomorphisms (17.2) may be applied to geometric motives like $M(X)$, X any \mathbb{C} -scheme of finite type (yielding $\mathrm{LAlb}(X)$), but also $M(X)^*(n)[2n]$ and $M^c(X)$ yielding $\mathrm{LAlb}^*(X)$ and $\mathrm{LAlb}^c(X)$ respectively. We thus get the following corollary, overlapping Deligne's conjecture (see 17.1.2 and 17.1.3 for the notation):

17.3.2. Corollary. *Let X be an n -dimensional complex algebraic variety. The mixed Hodge structures $H^i(X, \mathbb{Q})^{\leq 1}$, $H^{2n-i}(X, \mathbb{Q}(n))_{\leq 1}$ and $H_i^c(X, \mathbb{Q})_{\leq 1}$ induced by the Betti cohomology mixed Hodge structure and*

the Borel-Moore homology mixed Hodge structure admit a purely algebraic construction provided by the previously explained isomorphisms

$$\begin{aligned} R_1^{\text{Hodge}}(\mathbf{R}^i\text{Pic}(X)) &\xrightarrow{\sim} H^i(X, \mathbb{Q})^{\leq 1} \\ H^{2n-i}(X, \mathbb{Q}(n))_{\leq 1} &\xrightarrow{\sim} R_1^{\text{Hodge}}(\mathbf{L}_i\text{Alb}^*(X)) \\ H_i^c(X, \mathbb{Q})_{\leq 1} &\xrightarrow{\sim} R_1^{\text{Hodge}}(\mathbf{L}_i\text{Alb}^c(X)) \end{aligned}$$

of mixed Hodge structures.

17.3.3. Remark. Deligne's conjecture [25, (10.4.1)] concerns three types of Hodge structures of level ≤ 1 for X of dimension $\leq N$: $I(H^n(X, \mathbb{Z}))$, $II_n(H^n(X, \mathbb{Z}))$ ($n \leq N$) and $II_N(H^n(X, \mathbb{Z}))$ ($n \geq N$). Corollary 17.3.2 covers the first and last (compare Remark 17.1.3), but not the second in general. The issue for II_n and II_N is that the motive $M(X)^*(n)[2n]$ is effective for $n \geq \dim X$ by Lemma 9.2.1, but not for $n < \dim X$ in general. Indeed, if $M(X)^*(n)[2n]$ is effective, then it is isomorphic in $\text{DM}_{-}^{\text{eff}} \otimes \mathbb{Q}$ to $\underline{\text{Hom}}_{\text{eff}}(M(X), \mathbb{Z}(n)[2n])$ for formal reasons, and therefore the latter is a geometric motive. But this is false *e.g.* for $n = 2$ and X a suitable smooth projective 3-fold, see [2].

Suppose that the motivic t -structure exists on $\text{DM}_{\text{gm}}^{\text{eff}} \otimes \mathbb{Q}$. By a recent result of Beilinson, this implies Grothendieck's standard conjecture B. For any X , let us then write $M_i(X)$ for the i -th t -homology of $M(X)$. If X is smooth projective, by Poincaré duality and Conjecture B we find that

$$\underline{\text{Hom}}_{\text{eff}}(M_n(X), \mathbb{Z}(n)) \simeq M_n(X)$$

is geometric. By blow-up induction, this then implies that the motive $\underline{\text{Hom}}_{\text{eff}}(M_n(X), \mathbb{Z}(n))$ is geometric for any X of finite type, and we obtain the remaining part of Deligne's conjecture.

A less expensive approach could be to use the extension of $\text{LAlb}^{\mathbb{Q}}$ to $\text{DM}^{\text{eff}} \otimes \mathbb{Q}$ constructed in [4], provided one can similarly extend the Hodge realisation functor to $\text{DM}^{\text{eff}} \otimes \mathbb{Q}$.

17.4. Deligne's Hodge realisation functor. In Deligne's construction, the *integrally defined* Hodge realization

$$T^{\text{Hodge}}(M) := (T_{\mathbb{Z}}(M), W_*, F^*)$$

of a 1-motive (with torsion) M over $k = \mathbb{C}$ (see [7, §1] and [25, 10.1.3]) is obtained as follows. The finitely generated abelian group $T_{\mathbb{Z}}(M)$ is given by the pull-back of $u : L \rightarrow G$ along $\exp : \text{Lie}(G) \rightarrow G$, W_* is the

weight filtration

$$W_i T(M) := \begin{cases} T_{\mathbb{Z}}(M) \otimes \mathbb{Q} & i \geq 0 \\ H_1(G, \mathbb{Q}) & i = -1 \\ H_1(T, \mathbb{Q}) & i = -2 \\ 0 & i \leq -3 \end{cases}$$

and F^* is defined by $F^0(T_{\mathbb{Z}}(M) \otimes \mathbb{C}) := \text{Ker}(T_{\mathbb{Z}}(M) \otimes \mathbb{C} \rightarrow \text{Lie}(G))$. We have that $T_{\mathbb{Z}}(M)$, W_* and F^0 are independent of the representative of M . Thus $T^{\text{Hodge}}(M)$ is a mixed \mathbb{Z} -Hodge structure such that gr_i^W is polarizable. We have $\text{gr}_0^W T^{\text{Hodge}}(M) \cong L \otimes \mathbb{Q}$, $\text{gr}_{-1}^W T^{\text{Hodge}}(M) \cong H_1(A, \mathbb{Q})$ and $\text{gr}_{-2}^W T^{\text{Hodge}}(M) \cong H_1(T, \mathbb{Q})$ as pure polarizable \mathbb{Q} -Hodge structures.

Let $\text{MHS}_1^{\mathbb{Z}}$ be the category of mixed \mathbb{Z} -Hodge structure with possibly non zero Hodge numbers in the set $\{(0, 0), (0, -1), (-1, 0), (-1, -1)\}$ such that gr_i^W is polarizable. We have $\text{MHS}_1^{\mathbb{Z}} \otimes \mathbb{Q} = \text{MHS}_{(1)}$.

The functor T^{Hodge} is the *covariant* Deligne's Hodge realization

$$T^{\text{Hodge}} : {}^t\mathcal{M}_1(\mathbb{C}) \xrightarrow{\simeq} \text{MHS}_1^{\mathbb{Z}}$$

which is an equivalence of abelian categories by [7, Prop. 1.5]. It induces an equivalence

$$T_{\mathbb{Q}}^{\text{Hodge}} : D^b(\mathcal{M}_1(\mathbb{C}) \otimes \mathbb{Q}) \xrightarrow{\simeq} D^b(\text{MHS}_{(1)}).$$

17.4.1. Remark. We don't prove here that the functor R_1^{Hodge} of Theorem 17.2.1 is isomorphic to $T_{\mathbb{Q}}^{\text{Hodge}}$. By Theorem 16.2.2, there exists at most one such natural isomorphism. Its existence would follow from a validation of Remark 16.2.3 in the Hodge context; V. Vologodsky actually gives a construction of it in a recent preprint [89]. This would reprove Theorem 17.2.1 (using [25, 10.1.3]) and yield a naturally commutative diagram

$$\begin{array}{ccc} D^b(\text{MHS}_{(1)}) & \xrightleftharpoons[(-)_{\leq 1}]{\iota} & D^b(\text{MHS}^{\text{eff}}) \\ T_{\mathbb{Q}}^{\text{Hodge}} \uparrow & & \uparrow R^{\text{Hodge}} \\ D^b(\mathcal{M}_1(\mathbb{C}) \otimes \mathbb{Q}) & \xrightleftharpoons[\text{LAlb}^{\mathbb{Q}}]{\text{Tot}} & \text{DM}_{\text{gm}}^{\text{eff}}(\mathbb{C}) \otimes \mathbb{Q}. \end{array}$$

18. THE MIXED REALISATION

We consider here the other part of Deligne's conjecture:

Les morphismes

$$\begin{aligned} T_\ell(I(H^n(X, \mathbb{Z}))) &\rightarrow (H^n(X, \mathbb{Z}_\ell)/\text{torsion})(1) \\ T_\ell(II_n(H^n(X, \mathbb{Z}))) &\leftarrow H^n(X, \mathbb{Z}_\ell)(n) \quad (\text{pour } n \leq N) \\ T_\ell(II_N(H^n(X, \mathbb{Z}))) &\leftarrow H^n(X, \mathbb{Z}_\ell)(N) \quad (\text{pour } n \geq N) \end{aligned}$$

et leurs analogues en cohomologie de De Rham devraient aussi admettre une définition purement algébrique.

Huber's Hodge realisation is only a part of her construction of a functor from $\mathrm{DM}_{\mathrm{gm}}(k, \mathbb{Q})$ (k a finitely generated field over \mathbb{Q}) to her category of mixed realisations. We exploit this much richer structure to give a proof of the above conjecture, rationally and excluding the second case (see Remark 17.3.3 for the latter).

18.1. LAlb^τ for mixed realisations. Let k be a finitely generated extension of \mathbb{Q} . Recall from [40, Def. 11.1.1] Huber's category \mathcal{MR} of mixed realisations. An object $A \in \mathcal{MR}$ is a collection

$$(A_{\mathrm{DR}}, A_\ell, A_{\sigma, \ell}, A_\sigma, A_{\sigma, \mathbb{C}}; I_{\mathrm{DR}, \sigma}, I_{\sigma, \mathbb{C}}, I_{\bar{\sigma}, \ell}, I_{\ell, \sigma})_{\ell \in \mathcal{P}, \sigma \in S}$$

where \mathcal{P} is the set of prime numbers, S is the set of embeddings $k \hookrightarrow \mathbb{C}$ and

- A_{DR} is a [finite dimensional] bifiltered k -vector space.
- A_ℓ is a filtered \mathbb{Q}_ℓ -adic representation of $G_k := \mathrm{Gal}(\bar{k}/k)$; it is assumed to be constructible (*i.e.* unramified over some model of finite type of k/\mathbb{Z}) and that the filtration is a weight filtration.
- $A_{\sigma, \ell}$ is a filtered \mathbb{Q}_ℓ -vector space.
- A_σ is a filtered \mathbb{Q} -vector space.
- $A_{\sigma, \mathbb{C}}$ is a filtered \mathbb{C} -vector space.
- $I_{\mathrm{DR}, \sigma} : A_{\mathrm{DR}} \otimes_\sigma \mathbb{C} \rightarrow A_{\sigma, \mathbb{C}}$ is a filtered isomorphism.
- $I_{\sigma, \mathbb{C}} : A_\sigma \otimes_{\mathbb{Q}} \mathbb{C} \rightarrow A_{\sigma, \mathbb{C}}$ is a filtered isomorphism.
- $I_{\bar{\sigma}, \ell} : A_\sigma \otimes_{\mathbb{Q}} \mathbb{Q}_\ell \rightarrow A_{\sigma, \ell}$ is a filtered isomorphism.
- $I_{\ell, \sigma} : A_\ell \rightarrow A_{\sigma, \ell}$ is a filtered isomorphism.

Additionally, it is required that the systems $(A_\sigma, A_{\mathrm{DR}}, A_{\sigma, \mathbb{C}}, I_{\mathrm{DR}, \sigma}, I_{\sigma, \mathbb{C}})$ define mixed Hodge structures.

Morphisms in \mathcal{MR} are defined in the obvious way.

This is a refinement of the category defined by Jannsen [46, Ch. 1] and Deligne [28]: the refinement is that on the ℓ -adic components of an object, the filtration is required to be a weight filtration. Huber also defined a triangulated t -category $D_{\mathcal{MR}}$, with heart \mathcal{MR} (*loc. cit.* Def. 11.1.3); it comes with a canonical functor (*loc. cit.* p. 94)

$$D^b(\mathcal{MR}) \rightarrow D_{\mathcal{MR}}$$

which is the identity on the hearts.

These categories are not sufficient for our purposes: instead, we shall use the categories \mathcal{MR}^P and $D_{\mathcal{MR}^P}$ of [40, Def. 21.1.1 and 21.1.3], which restrict the Hodge components to polarizable mixed Hodge structures: by definition, \mathcal{MR}^P is a full subcategory of \mathcal{MR} . There is the same formalism as above (*loc. cit.* Prop. 21.1.4), which maps to the previous one.

We need an “effective” full subcategory $\mathcal{MR}^{P,\text{eff}} \subset \mathcal{MR}^P$, and a corresponding effective triangulated category $D_{\mathcal{MR}^P}^{\text{eff}} \subset D_{\mathcal{MR}^P}$:

18.1.1. Definition. a) An object $X \in \mathcal{MR}^P$ is in $\mathcal{MR}^{P,\text{eff}}$ if

- (1) The de Rham component X_{DR} verifies $F^{0,0}X_{\text{DR}} = X_{\text{DR}}$ (non-positive Hodge numbers).
- (2) The eigenvalues of (arithmetic) Frobenius acting on the ℓ -adic components X_ℓ are algebraic integers.

b) An object $C \in D_{\mathcal{MR}^P}$ is in $D_{\mathcal{MR}^P}^{\text{eff}}$ if $H_i^t(C) \in \mathcal{MR}^{P,\text{eff}}$ for all $i \in \mathbb{Z}$, where H_*^t is the homology respective to the canonical t -structure of $D_{\mathcal{MR}^P}$.

In this way, the above functor refines to

$$D^b(\mathcal{MR}^{P,\text{eff}}) \rightarrow D_{\mathcal{MR}^P}^{\text{eff}}.$$

By Remark E.21.3 and [40, Prop. 11.1.5] (adapted to \mathcal{MR}^P), \mathcal{MR}^P and $D_{\mathcal{MR}^P}$ enjoy weights filtrations in the sense of Definitions E.7.2 and E.17.1, which are compatible in the sense of Proposition E.21.2. Moreover, the categories of pure weights \mathcal{MR}_n^P are semi-simple [40, Prop. 21.1.2], which is the main point of passing from \mathcal{MR} to \mathcal{MR}^P . These weight filtrations induce compatible weight filtrations on $\mathcal{B} = \mathcal{MR}^{P,\text{eff}}$ and $\mathcal{T} = D_{\mathcal{MR}^P}^{\text{eff}}$. We take for $\mathcal{B}_{\mathbb{L}}$ the full subcategory of \mathcal{MR}^P consisting of objects of the form $R_{\mathcal{MR}^P}(A \otimes \mathbf{T})$, where A is a (pure) Artin motive and \mathbf{T} is the Tate motive. Here $R_{\mathcal{MR}^P}$ is the functor defined in [40, Th. 20.2.3] on Chow motives: in *loc. cit.* it takes values in \mathcal{MR} but its essential image actually lands into \mathcal{MR}^P .

With the notation of §15.3, $\mathcal{B}_{(1)} = \mathcal{MR}_{(1)}^P$ is the full subcategory of $\mathcal{MR}^{P,\text{eff}}$ of objects X such that the possibly non zero Hodge numbers of X_{DR} are in the set $\{(0, 0), (0, -1), (-1, 0), (-1, -1)\}$. (Indeed, this condition forces the relevant conditions on the ℓ -adic components.)

We have seen that the full subcategory \mathcal{MR}_{-2}^P of pure objects of weight -2 is semi-simple. As a special case of Proposition 15.3.4, we therefore have:

18.1.2. Proposition. *The full embeddings $\iota : \mathcal{MR}_{(1)}^P \hookrightarrow \mathcal{MR}^{P,\text{eff}}$ and $\iota : (D_{\mathcal{MR}^P}^{\text{eff}})_{(1)} \hookrightarrow D_{\mathcal{MR}^P}^{\text{eff}}$ have left adjoints $\text{Alb}^{\mathcal{MR}^P}$ and $\text{LAlb}^{\mathcal{MR}^P}$. \square*

18.1.3. **Notation.** For simplicity we shall write $(-)_{\leq 1}$ for $\text{Alb}^{\mathcal{MR}^P}$.

We shall need the following lemma:

18.1.4. **Lemma.** *Let X be smooth projective over k . Then the map*

$$\text{Pic}(X) \otimes \mathbb{Q} \rightarrow \text{Hom}_{\mathcal{MR}^P}(\mathbb{Q}, H_{\mathcal{MR}^P}^2(X)(1))$$

given by the realisation functor $R_{\mathcal{MR}^P}$ induces an isomorphism

$$\text{NS}(X) \otimes \mathbb{Q} \xrightarrow{\sim} \text{Hom}_{\mathcal{MR}^P}(\mathbb{Q}, H_{\mathcal{MR}^P}^2(X)(1)).$$

Proof. Clearly the given map factors through $\text{NS}(X) \otimes \mathbb{Q}$, since all its components do, and the resulting map is injective because its Betti components are. To prove surjectivity, choose an embedding $\bar{\sigma} : \bar{k} \hookrightarrow \mathbb{C}$. For any finite extension E/k we have a commutative diagram

$$\begin{array}{ccc} \text{NS}(X_E) \otimes \mathbb{Q} & \longrightarrow & \text{Hom}_{\mathcal{MR}^P(E)}(\mathbb{Q}, H_{\mathcal{MR}^P(E)}^2(X_E)(1)) \\ \downarrow & & \bar{\sigma}|_E \downarrow \\ \text{NS}(\bar{X}) \otimes \mathbb{Q} & \longrightarrow & H_{\text{Hodge}}^2(X_{\mathbb{C}}, \mathbb{Q}(1))^{(1,1)} \end{array}$$

in which all maps are injective ($\bar{X} = X \otimes_k \bar{k}$). The bottom row is an isomorphism by the (1,1) theorem and the isomorphism $\text{NS}(\bar{X}) \otimes \mathbb{Q} \xrightarrow{\sim} \text{NS}(X_{\mathbb{C}}) \otimes \mathbb{Q}$. Passing to the limit, we get a commutative diagram

$$\begin{array}{ccc} \text{NS}(\bar{X}) \otimes \mathbb{Q} & \longrightarrow & \varinjlim_E \text{Hom}_{\mathcal{MR}^P(E)}(\mathbb{Q}, H_{\mathcal{MR}^P(E)}^2(X_E)(1)) \\ \parallel & & \bar{\sigma} \downarrow \\ \text{NS}(\bar{X}) \otimes \mathbb{Q} & \xrightarrow{\sim} & H_{\text{Hodge}}^2(X_{\mathbb{C}}, \mathbb{Q}(1))^{(1,1)} \end{array}$$

of injections, which forces the top horizontal map to be bijective. To conclude, we use the commutative diagram

$$\begin{array}{ccc} \text{NS}(X) \otimes \mathbb{Q} & \longrightarrow & \text{Hom}_{\mathcal{MR}^P}(\mathbb{Q}, H_{\mathcal{MR}^P}^2(X)(1)) \\ \downarrow \wr & & \downarrow \wr \\ (\text{NS}(\bar{X}) \otimes \mathbb{Q})^{G_k} & \longrightarrow & \varinjlim_E \text{Hom}_{\mathcal{MR}^P(E)}(\mathbb{Q}, H_{\mathcal{MR}^P(E)}^2(X_E)(1))^{G_k} \end{array}$$

where the vertical maps are isomorphisms by the usual transfer argument. \square

18.2. **Huber's mixed realisation functor.** In [42, Th. 2.3.3 and following remark], Annette Huber defines a (contravariant) realisation functor:

$$R_{\mathcal{MR}^P} : \text{DM}_{\text{gm}}(k, \mathbb{Q})^{\text{op}} \rightarrow D_{\mathcal{MR}^P}.$$

The categories \mathcal{MR}^P and $D_{\mathcal{MR}^P}$ carry a duality. As in the previous section, we define a covariant realisation functor

$$R^{\mathcal{MR}^P} : \mathrm{DM}_{\mathrm{gm}}(k, \mathbb{Q}) \rightarrow D_{\mathcal{MR}^P}$$

as the composite of $R_{\mathcal{MR}^P}$ with this duality. This induces a functor

$$R^{\mathcal{MR}^P} : \mathrm{DM}_{\mathrm{gm}}^{\mathrm{eff}}(k, \mathbb{Q}) \rightarrow D_{\mathcal{MR}^P}^{\mathrm{eff}}.$$

Since $R_{\mathcal{MR}^P} \Phi : \mathrm{Chow}^{\mathrm{eff}} \otimes \mathbb{Q} \rightarrow D_{\mathcal{MR}^P}$ is by construction isomorphic to the mixed realisation functor $X \mapsto R_{\mathcal{MR}^P}(X)$ collecting all individual realisations of X with their comparison isomorphisms [40, Th. 20.2.3], the conditions of Hypothesis 15.4.1 are verified, as well as Condition (W) of Proposition 15.4.2. This proposition then shows that $R^{\mathcal{MR}^P} \mathrm{Tot}$ defines an exact functor

$$R_1^{\mathcal{MR}^P} : \mathcal{M}_1(k) \otimes \mathbb{Q} \rightarrow \mathcal{MR}_{(1)}^P.$$

18.3. Deligne's conjecture. From Theorem 17.2.1 and (15.4) we get, for any $M \in \mathrm{DM}_{\mathrm{gm}}^{\mathrm{eff}}(k) \otimes \mathbb{Q}$, a natural map

$$(18.1) \quad R^{\mathcal{MR}^P}(M)_{\leq 1} \rightarrow R_1^{\mathcal{MR}^P} \mathrm{LAlb}(M).$$

Taking homology of both sides, we get comparison maps

$$(18.2) \quad H_i(R^{\mathcal{MR}^P}(M))_{\leq 1} \rightarrow R_1^{\mathcal{MR}^P} \mathrm{L}_i \mathrm{Alb}(M)$$

for all $i \in \mathbb{Z}$. From Theorem 15.5.5 and Lemma 18.1.4, we now get:

18.3.1. Theorem. *The maps (18.1) and (18.2) are isomorphisms for all motives M . \square*

Let us develop this theorem in view of the definition of \mathcal{MR}^P . For this, we shall write R^ℓ , R^{DR} , R^σ for the ℓ -adic, de Rham and (relative to σ) Betti components of $R^{\mathcal{MR}^P}$, and similarly $R_1^\%$, $H_i^\% = H_i \circ R^\%$ ($\% \in \{\mathrm{DR}, \ell, \sigma\}$).

18.3.2. Corollary. *Let $M \in \mathrm{DM}_{\mathrm{gm}}^{\mathrm{eff}}(k, \mathbb{Q})$. Then:*

a) One has isomorphisms

$$\begin{aligned} H_i^\ell(M)_{\leq 1} &\xrightarrow{\sim} R_1^\ell \mathrm{L}_i \mathrm{Alb}(M) \\ H_i^{\mathrm{DR}}(M)_{\leq 1} &\xrightarrow{\sim} R_1^{\mathrm{DR}} \mathrm{L}_i \mathrm{Alb}(M) \\ H_i^\sigma(M)_{\leq 1} &\xrightarrow{\sim} R_1^\sigma \mathrm{L}_i \mathrm{Alb}(M). \end{aligned}$$

The last two isomorphisms yield the isomorphism of Hodge structures from Theorem 17.3.1, and the first one is an isomorphism of constructible Galois representations with weight filtrations.

b) *The diagrams*

$$\begin{array}{ccc}
H_i^\sigma(M)_{\leq 1} \otimes_{\mathbb{Q}} \mathbb{Q}_\ell & \xrightarrow{\sim} & R_1^\sigma L_i \text{Alb}(M) \otimes_{\mathbb{Q}} \mathbb{Q}_\ell \\
I_{\ell, \sigma}^{-1} \circ I_{\sigma, \ell} \downarrow & & I_{\ell, \sigma}^{-1} \circ I_{\sigma, \ell} \downarrow \\
H_i^\ell(M)_{\leq 1} & \xrightarrow{\sim} & R_1^\ell L_i \text{Alb}(M) \\
\\
H_i^\sigma(M)_{\leq 1} \otimes_{\mathbb{Q}} \mathbb{C} & \xrightarrow{\sim} & R_1^\sigma L_i \text{Alb}(M) \otimes_{\mathbb{Q}} \mathbb{C} \\
I_{\text{DR}, \sigma}^{-1} \circ I_{\sigma, \mathbb{C}} \downarrow & & I_{\text{DR}, \sigma}^{-1} \circ I_{\sigma, \mathbb{C}} \downarrow \\
H_i^{\text{DR}}(M)_{\leq 1} \otimes_k \mathbb{C} & \xrightarrow{\sim} & R_1^{\text{DR}} L_i \text{Alb}(M) \otimes_k \mathbb{C}
\end{array}$$

commute.

Corollary 18.3.2 partly extends to $\text{DM}_{\text{gm}}^{\text{eff}}(\mathbb{C}, \mathbb{Q})$. On the one hand, one has

$$2 - \varinjlim_{k \text{ f.g.}} \text{DM}_{\text{gm}}^{\text{eff}}(k, \mathbb{Q}) \xrightarrow{\sim} \text{DM}_{\text{gm}}^{\text{eff}}(\mathbb{C}, \mathbb{Q})$$

by [44, Prop. 4.16]. On the other hand, de Rham cohomology commutes with base change and ℓ -adic cohomology is invariant under algebraically closed extensions. This yields:

18.3.3. Corollary. *Let $M \in \text{DM}_{\text{gm}}^{\text{eff}}(\mathbb{C}, \mathbb{Q})$. Then:*

a) *The comparison isomorphisms extend to comparison isomorphisms of realisations:*

$$\begin{aligned}
R^B(M) \otimes_{\mathbb{Q}} \mathbb{Q}_\ell & \xrightarrow{I_\ell} R^\ell(M) \\
R^B(M) \otimes_{\mathbb{Q}} \mathbb{C} & \xrightarrow{I_{\text{DR}}} R^{\text{DR}}(M).
\end{aligned}$$

b) *There exist vector spaces*

$$H_i^\ell(M)_{\leq 1}, H_i^{\text{DR}}(M)_{\leq 1}, H_i^B(M)_{\leq 1}$$

quotients of the ℓ -adic, de Rham and Betti realisations of M , and functorially attached to M .

c) *One has isomorphisms*

$$\begin{aligned}
H_i^\ell(M)_{\leq 1} & \xrightarrow{\sim} R_1^\ell L_i \text{Alb}(M) \\
H_i^{\text{DR}}(M)_{\leq 1} & \xrightarrow{\sim} R_1^{\text{DR}} L_i \text{Alb}(M) \\
H_i^B(M)_{\leq 1} & \xrightarrow{\sim} R_1^\sigma L_i \text{Alb}(M).
\end{aligned}$$

The last two isomorphisms yield the isomorphism of Hodge structures from Theorem 17.3.1.

d) *The diagrams*

$$\begin{array}{ccc}
H_i^B(M)_{\leq 1} \otimes_{\mathbb{Q}} \mathbb{Q}_{\ell} & \xrightarrow{\sim} & R_1^B L_i \text{Alb}(M) \otimes_{\mathbb{Q}} \mathbb{Q}_{\ell} \\
I_{\ell} \downarrow & & I_{\ell} \downarrow \\
H_i^{\ell}(M)_{\leq 1} & \xrightarrow{\sim} & R_1^{\ell} L_i \text{Alb}(M) \\
H_i^B(M)_{\leq 1} \otimes_{\mathbb{Q}} \mathbb{C} & \xrightarrow{\sim} & R_1^B L_i \text{Alb}(M) \otimes_{\mathbb{Q}} \mathbb{C} \\
I_{\text{DR}} \downarrow & & I_{\text{DR}} \downarrow \\
H_i^{\text{DR}}(M)_{\leq 1} & \xrightarrow{\sim} & R_1^{\text{DR}} L_i \text{Alb}(M)
\end{array}$$

commute.

Applied to geometric motives like $M(X)$ or $M(X)^*(n)[2n]$, this gives a reasonable interpretation of the second part of Deligne's conjecture.

19. THE ℓ -ADIC REALISATION IN POSITIVE CHARACTERISTIC

In this section, k is a finitely generated field and ℓ is a prime number different from $\text{char } k$. From §19.3 onwards, we shall assume $\text{char } k > 0$.

19.1. The tame derived category of ℓ -adic sheaves. We shall use an analogue of the categories considered by Huber [41]. Namely,

19.1.1. Definition. Let

$$\tilde{D}(k, \mathbb{Z}_{\ell}) := 2 - \varinjlim D(S, \mathbb{Z}_{\ell})$$

where S runs through the regular models of k , of finite type over $\text{Spec } \mathbb{Z}$, and for such S , $D(S, \mathbb{Z}_{\ell})$ is the category defined by Ekedahl in [32], for the étale topos of S . Let

$$\tilde{D}_c^b(k, \mathbb{Z}_{\ell}) := 2 - \varinjlim D_c^b(S, \mathbb{Z}_{\ell})$$

where $D_c^b(S, \mathbb{Z}_{\ell})$ is the thick subcategory of $D(S, \mathbb{Z}_{\ell})$ consisting of those complexes whose cohomology sheaves are constructible. Finally,

$$\tilde{D}_m^b(k, \mathbb{Q}_{\ell}) := 2 - \varinjlim D_m^b(S, \mathbb{Q}_{\ell})$$

where $D_m^b(S, \mathbb{Q}_{\ell})$ is the thick subcategory of $D_c^b(S, \mathbb{Q}_{\ell})$ consisting of those complexes whose cohomology sheaves are mixed in the sense of [27, (1.2.2)].

Recall [32, Thm. 4.7] where the following pairings are constructed

$$\begin{aligned}
(19.1) \quad & \overset{L}{\otimes} : D^-(S, \mathbb{Z}_{\ell}) \times D(S, \mathbb{Z}_{\ell}) \rightarrow D(S, \mathbb{Z}_{\ell}) \\
& \text{RHom} : D(S, \mathbb{Z}_{\ell})^{\text{op}} \times D^+(S, \mathbb{Z}_{\ell}) \rightarrow D(S, \mathbb{Z}_{\ell})
\end{aligned}$$

From these we get similar pairings

$$(19.2) \quad \begin{aligned} \overset{L}{\otimes} : \tilde{D}^-(k, \mathbb{Z}_\ell) \times \tilde{D}(k, \mathbb{Z}_\ell) &\rightarrow \tilde{D}(k, \mathbb{Z}_\ell) \\ \mathrm{RHom} : \tilde{D}(k, \mathbb{Z}_\ell)^{\mathrm{op}} \times \tilde{D}^+(k, \mathbb{Z}_\ell) &\rightarrow \tilde{D}(k, \mathbb{Z}_\ell) \end{aligned}$$

19.1.2. Remark. By Deligne's finiteness theorems in étale cohomology [SGA4 1/2, Th. finitude], the categories $D_c^b(S, \mathbb{Z}_\ell)$ and $D_m^b(S, \mathbb{Z}_\ell)$ are equivalent to those considered in [10].

The category $\tilde{D}_m^b(k, \mathbb{Q}_\ell)$ enjoys a t -structure which is the 2-colimit of the natural t -structures of $D_m^b(S, \mathbb{Q}_\ell)$, with hearts the categories $\mathrm{Mix}(S, \mathbb{Q}_\ell)$ of mixed \mathbb{Q}_ℓ -sheaves on S .

19.1.3. Definition. We write $\mathrm{Mix}(k, \mathbb{Q}_\ell)$ for the 2-colimit of the $\mathrm{Mix}(S, \mathbb{Q}_\ell)$: this is the heart of $\tilde{D}_m^b(k, \mathbb{Q}_\ell)$.

19.1.4. Remarks. 1) For any mixed sheaf \mathcal{F} on S , the restriction of \mathcal{F} to a suitable open set is lisse, and if \mathcal{F} and \mathcal{G} are lisse over S , the map

$$\mathrm{Hom}_S(\mathcal{F}, \mathcal{G}) \rightarrow \mathrm{Hom}_U(\mathcal{F}, \mathcal{G})$$

is bijective for any open subset $U \subseteq S$, as one sees by interpreting \mathcal{F} and \mathcal{G} as ℓ -adic representations of $\pi_1(S)$ [SGA5, VI]. Hence the functor

$$2 - \varinjlim \mathrm{Mix}(S, \mathbb{Q}_\ell) \rightarrow \mathrm{Mix}(k, \mathbb{Q}_\ell)$$

is fully faithful, and objects of $\mathrm{Mix}(k, \mathbb{Q}_\ell)$ may be identified with “generically unramified” ℓ -adic representations of G_k , the absolute Galois group of k .

2) Let $\mathrm{Perv}(S, \mathbb{Q}_\ell)$ be the category of perverse sheaves over S , for the middle perversity. By definition of perverse sheaves, we also have an equivalence

$$2 - \varinjlim \mathrm{Perv}(S, \mathbb{Q}_\ell) \xrightarrow{\sim} \mathrm{Mix}(k, \mathbb{Q}_\ell)[\dim S]$$

inside $\tilde{D}_m^b(k, \mathbb{Q}_\ell)$.

3) The definition of $D^-(S, \mathbb{Z}_\ell)$ yields canonical t -exact functors

$$D^b(\mathrm{Mix}(S, \mathbb{Q}_\ell)) \rightarrow D_m^b(S, \mathbb{Q}_\ell)$$

hence a canonical t -exact functor

$$D^b(\mathrm{Mix}(k, \mathbb{Q}_\ell)) \rightarrow \tilde{D}_m^b(k, \mathbb{Q}_\ell)$$

which is the identity on the hearts.

19.2. ℓ -adic realisations for DM. Let $\text{EST}(S, R)$ be the category of étale sheaves of R -modules with transfers over S (using the category Cor_S of [61, App. 1A]). If $R = \mathbb{Z}$ we drop it from the notation. Since $\text{EST}(S)$ has enough flat objects, namely the objects $\mathbb{Z}_{\text{tr}}(X)$ for X smooth over S , we get natural functors

$${}^L\otimes \mathbb{Z}/\ell^\nu : D(\text{EST}(S)) \rightarrow D(\text{EST}(S, \mathbb{Z}/\ell^\nu)).$$

These functors are clearly compatible. Let $R^{(\nu)} : \text{EST}(S, \mathbb{Z}/\ell^\nu) \rightarrow \text{Shv}(S_{\text{ét}}, \mathbb{Z}/\ell^\nu)$ be the functor that forgets transfers followed by the restriction to the small étale site of S . Then $R^{(\nu)}$ induces two functors:

$$R^{(\nu)} : D(\text{EST}(S, \mathbb{Z}/\ell^\nu)) \rightarrow D(S, \mathbb{Z}/\ell^\nu)$$

$$R_{(\nu)} : D(\text{EST}(S, \mathbb{Z}/\ell^\nu))^{\text{op}} \rightarrow D(S, \mathbb{Z}/\ell^\nu)$$

where $R_{(\nu)}(-) := \text{RHom}(R^{(\nu)}(-), \mathbb{Z}/\ell^\nu)$. For $K \in \text{EST}(S)$ we thus have an inverse system of pairings

$$R^{(\nu)}(K \otimes \mathbb{Z}/\ell^\nu) \otimes R_{(\nu)}(K \otimes \mathbb{Z}/\ell^\nu) \rightarrow \mathbb{Z}/\ell^\nu$$

which induce a pair of dual triangulated functors (relying on (19.1))

$$R^\ell : D(\text{EST}(S)) \rightarrow D(S, \mathbb{Z}_\ell)$$

$$R_\ell : D(\text{EST}(S))^{\text{op}} \rightarrow D(S, \mathbb{Z}_\ell).$$

For $f : X \rightarrow S$ smooth, we have that

$$(19.3) \quad R_\ell(\mathbb{Z}_{\text{tr}}(X)) \cong Rf_* \mathbb{Z}_\ell$$

by [61, Porism 6.24]¹⁵.

Recall that over the base S the category $\text{DM}^{\text{eff}}(S)$ has the same descriptions as done in [86] or [61, Lect. 14] over a field: see [88]. Namely, it may be defined as the localisation of the derived category of Nisnevich sheaves with transfers with respect to \mathbb{A}^1 -equivalences. Moreover, this localisation functor has a right adjoint so that it may be regarded as the full subcategory of \mathbb{A}^1 -local objects, *cf.* [4, 2.2.6].

The same picture works for étale sheaves with transfers, at least if $cd_l(S) < \infty$: let

$$L_{\mathbb{A}^1} : D(\text{EST}(S)) \rightarrow \text{DM}_{\text{ét}}^{\text{eff}}(S)$$

be the localisation of $D(\text{EST}(S))$ with respect to \mathbb{A}^1 -equivalences. Then $L_{\mathbb{A}^1}$ has a fully faithful right adjoint i , identifying $\text{DM}_{\text{ét}}^{\text{eff}}(S)$ with the full subcategory of \mathbb{A}^1 -local objects in $D(\text{EST}(S))$.

Since ℓ -adic cohomology is homotopy invariant for smooth schemes and the $\mathbb{Z}_{\text{tr}}(X)$ are dense in $D(\text{EST}(S))$, 19.3 implies that the functor

¹⁵The proof of loc. cit. is for $S =$ a field, but it works in general.

R_ℓ factors through $\mathrm{DM}_{\text{ét}}^{\text{eff}}(S)$ via $L_{\mathbb{A}^1}$. On the other hand, we can restrict R^ℓ to $\mathrm{DM}_{\text{ét}}^{\text{eff}}(S)$. By adjunction, we end up with a pair of dual triangulated functors

$$\begin{aligned} R^\ell &: \mathrm{DM}_{\text{ét}}^{\text{eff}}(S) \rightarrow D(S, \mathbb{Z}_\ell) \\ R_\ell &: \mathrm{DM}_{\text{ét}}^{\text{eff}}(S)^{\text{op}} \rightarrow D(S, \mathbb{Z}_\ell). \end{aligned}$$

Recall the category $\mathrm{DM}_{\text{gm}}^{\text{eff}}(S)$, constructed over a base exactly as over a field except that the Mayer-Vietoris relations for the Zariski topology are replaced by Nisnevich excision relations [44, Def. 1.14]. In his thesis, Ivorra also constructed a functor $\mathrm{DM}_{\text{gm}}^{\text{eff}}(S) \rightarrow \mathrm{DM}_{\text{Nis}}^{\text{eff}}(S)$ extending Voevodsky's functor for base fields (see [45]). Using it and the change-of-topology functor $\mathrm{DM}_{\text{Nis}}^{\text{eff}}(S) \rightarrow \mathrm{DM}_{\text{ét}}^{\text{eff}}(S)$, we can restrict R^ℓ and R_ℓ to $\mathrm{DM}_{\text{gm}}^{\text{eff}}(S)$.

Ivorra also constructs a realisation functor $\mathrm{DM}_{\text{gm}}^{\text{eff}}(S)^{\text{op}} \rightarrow D^+(S, \mathbb{Z}_\ell)$ in [44, Th. 4.3 (b)]. Using (19.3), we can show:

19.2.1. Lemma. *R_ℓ coincides with Ivorra's functor.* \square

Proof. Write R_l^I for Ivorra's functor. We construct a pairing

$$(19.4) \quad R_l^I(M) \overset{L}{\otimes} R^l(M) \rightarrow \mathbb{Z}_l$$

natural in $M \in \mathrm{DM}_{\text{gm}}^{\text{eff}}(S)$. Since R_l^I is constructed by descent through the natural functor $C^b(\mathrm{Cor}_S) \rightarrow \mathrm{DM}_{\text{gm}}^{\text{eff}}(S)$, where Cor_S is as in [61, App. 1A], it suffices to construct (19.4) after composition with this functor.

If $C = (X_n, d_n)$ is a bounded complex of finite correspondences, by [44, p. 656], $R_l^I(C)$ is the image in $D^+(S, \mathbb{Z}/l^*)$ of the total complex associated to the double complex

$$(\underline{R}_l(X_n), \underline{R}_l(d_n))$$

where, for $\pi : X \rightarrow S$ smooth, $\underline{R}_l(X) \in C^+(S, \mathbb{Z}/l^*)$ is the direct image under π of the Godement resolution of \mathbb{Z}/l^* over $X_{\text{ét}}$, which is canonically provided with transfers by [44, §3.2].

On the other hand, $R^l(C)$ is by definition the image of the complex

$$(\mathbb{Z}_{\text{tr}}(X_n) \otimes \mathbb{Z}/l^*, \mathbb{Z}_{\text{tr}}(d_n) \otimes \mathbb{Z}/l^*).$$

We start with the case where C is concentrated in degree 0, *i.e.* $C = X[0]$. Let $\pi : X \rightarrow S$ be the structural morphism. Let \mathcal{F} be an étale sheaf with transfers. The canonical isomorphism [61, Cor. 6.3]

$$\mathcal{F}(X) = \mathrm{Hom}_{\mathrm{EST}(S)}(\mathbb{Z}_{\text{tr}}(X), \mathcal{F})$$

provides, by adjunction and varying the base S , a canonical pairing of sheaves

$$\pi_* \pi^* \mathcal{F} \otimes \mathbb{Z}_{\text{tr}}(X) \rightarrow \mathcal{F}.$$

Since this is natural in \mathcal{F} , it naturally extends to a pairing of Godement resolutions

$$\pi_* Gd(\pi^* \mathcal{F}) \otimes \mathbb{Z}_{\text{tr}}(X) \rightarrow Gd(\mathcal{F}).$$

Applying this to $\mathcal{F} = \mathbb{Z}/l^*$, we get a pairing

$$\underline{R}_l(X) \otimes (\mathbb{Z}/l^*)_{\text{tr}}(X) \rightarrow \mathbb{Z}/l^*$$

in $C^+(S, \mathbb{Z}/l^*)$. By construction, this pairing is natural in X for the action of finite correspondences; therefore it extends to the general pairing (19.4) restricted to $C^b(\text{Cor}_S)$, hence finally a morphism of functors

$$R_l^I \Rightarrow R_l$$

on $\text{DM}_{\text{gm}}^{\text{eff}}(S)$.

This morphism is an isomorphism for $M = M(X)$, X smooth, as seen before; therefore it is an isomorphism for all M . \square

19.2.a. Functoriality. The construction of R^ℓ is contravariant in S . More precisely, for any morphism $f : T \rightarrow S$, we have a tautological natural isomorphism

$$R_T^\ell \circ f^* \simeq f^* \circ R_S^\ell.$$

From it, one gets a natural transformation, for $M \in \text{DM}_{\text{et}}^{\text{eff}}(S)$:

$$(19.5) \quad f^* R_\ell^S(M) = f^* \underline{\text{Hom}}_S(R_S^\ell M, \mathbb{Z}_\ell) \rightarrow \underline{\text{Hom}}_T(f^* R_S^\ell M, \mathbb{Z}_\ell) \\ \simeq \underline{\text{Hom}}_T(R_S^\ell f^* M, \mathbb{Z}_\ell) = R_\ell^T f^* M.$$

19.2.2. Proposition (*cf.* [44, Rem. 4.9]). *If f is smooth, (19.5) is a natural isomorphism.*

Proof. It suffices to test this on generators $M = M(X)$, with $\pi : X \rightarrow S$ smooth. By (19.3), (19.5) is then the base change map

$$f^* R\pi_* \mathbb{Z}_\ell \rightarrow R\pi'_* f'^* \mathbb{Z}_\ell$$

with $f' : X \times_S T \rightarrow X$, $\pi' : X \times_S T \rightarrow T$: it is an isomorphism by the smooth base change theorem [SGA4, Exp. XVI, Cor. 1.2]. \square

By [44, Prop. 4.16], if S runs through the smooth models of k over \mathbb{F}_p , the natural functor

$$2 - \varinjlim \text{DM}_{\text{gm}}^{\text{eff}}(S) \rightarrow \text{DM}_{\text{gm}}^{\text{eff}}(k)$$

is an equivalence of categories; hence, by Proposition 19.2.2, the R_S^ℓ and R_ℓ^S induce a pair of dual triangulated functors (relying on (19.2))

$$\begin{aligned} R^\ell : \mathrm{DM}_{\mathrm{gm}}^{\mathrm{eff}}(k) &\rightarrow \tilde{D}(k, \mathbb{Z}_\ell) \\ R_\ell : \mathrm{DM}_{\mathrm{gm}}^{\mathrm{eff}}(k)^{\mathrm{op}} &\rightarrow \tilde{D}(k, \mathbb{Z}_\ell). \end{aligned}$$

19.3. Weights and niveaux. By definition of a mixed sheaf, we can define a filtration $(\mathrm{Mix}(k, \mathbb{Q}_l)_{\leq n})_{n \in \mathbb{Z}}$ on $\mathrm{Mix}(k, \mathbb{Q}_l)$ in the sense of Definition E.7.1 as follows: $\mathcal{F} \in \mathrm{Mix}(k, \mathbb{Q}_l)$ is in $\mathrm{Mix}(k, \mathbb{Q}_l)_{\leq n}$ if and only if its punctual weights (on a suitable model of k) are all $\leq n$. This filtration is clearly exhaustive and separated.

This is not a weight filtration if $\mathrm{char} k = 0$: this fact was pointed out by J. Wildeshaus. More precisely, the inclusions

$$\iota_n : \mathrm{Mix}(k, \mathbb{Q}_l)_{\leq n} \hookrightarrow \mathrm{Mix}(k, \mathbb{Q}_l)_{\leq n+1}$$

do not have right adjoints: see [46, p. 90, Remark 6.8.4 i)]. Fortunately, weights are taken care of directly in Section 18 using the mixed realisations; for this reason, *we now assume* $\mathrm{char} k > 0$.

By Remarks 19.1.2 and 19.1.4 and by [27, (3.4.1)], the category $\mathrm{Mix}(k, \mathbb{Q}_\ell)$ of Definition 19.1.3 then enjoys a good theory of weights. Namely:

19.3.1. Proposition. *Suppose $\mathrm{char} k > 0$.*

- a) *The filtration $\mathrm{Mix}(k, \mathbb{Q}_l)_{\leq n}$ provides $\mathrm{Mix}(k, \mathbb{Q}_\ell)$ with a weight filtration in the sense of Definition E.7.2.*
- b) *The categories $\mathrm{Mix}(k, \mathbb{Q}_l)_n$ of objects of pure weight n are semi-simple.*

Proof. a) Consider the filtration W_n defined on mixed sheaves by [27, (3.4.1) (ii)]. By Remark E.7.8, it suffices to know that the W_n are exact, exhaustive and separated. The last two facts are clear, and the first follows from the strict compatibility with morphisms. b) follows from [27, (3.4.1) (iii)]. \square

The weight filtration we shall need on $\tilde{D}_m^b(k, \mathbb{Q}_\ell)$ is not the one considered in [10] and [41], but rather the one in Definition E.21.1, which coincides with the one introduced by S. Morel in [64, §3.1]:

19.3.2. Definition. An object $C \in \tilde{D}_m^b(k, \mathbb{Q}_\ell)$ is of *weight* $\leq w$ if the weights of $H^i(C)$ are $\leq w$ for all $i \in \mathbb{Z}$ (compare Remark 19.1.4 2)).

We also need a notion of effectivity:

19.3.3. Definition. a) An object $\mathcal{F} \in \mathrm{Mix}(k, \mathbb{Q}_\ell)$ is *effective* if the eigenvalues of *arithmetic* Frobenius elements acting on the stalk(s) of

(a finite level representative of) \mathcal{F} are algebraic integers. We denote by $\text{Mix}(k, \mathbb{Q}_\ell)^{\text{eff}}$ their full subcategory.

b) We denote by $\tilde{D}_m^b(k, \mathbb{Q}_\ell)^{\text{eff}}$ the full subcategory of $\tilde{D}_m^b(k, \mathbb{Q}_\ell)$ consisting of objects with cohomology in $\text{Mix}(k, \mathbb{Q}_\ell)^{\text{eff}}$.

Clearly, the abelian category $\text{Mix}(k, \mathbb{Q}_\ell)^{\text{eff}}$ is thick in $\text{Mix}(k, \mathbb{Q}_\ell)$, hence $\tilde{D}_m^b(k, \mathbb{Q}_\ell)^{\text{eff}}$ is a thick triangulated subcategory of $\tilde{D}_m^b(k, \mathbb{Q}_\ell)$, with heart $\text{Mix}(k, \mathbb{Q}_\ell)^{\text{eff}}$.

19.3.4. Proposition. R^ℓ verifies Hypotheses 15.4.1 and 15.5.2.

Proof. We first prove that R^ℓ maps $\text{DM}_{\text{gm}}^{\text{eff}}(k) \otimes \mathbb{Q}$ into $\tilde{D}_m^b(k, \mathbb{Q}_\ell)$ (this much does not need characteristic p). It suffices to prove that for any smooth model S of k , R^ℓ maps $\text{DM}_{\text{gm}}^{\text{eff}}(S) \otimes \mathbb{Q}$ into $D_m^b(S, \mathbb{Q}_\ell)$. Since this category is stable under duality, it suffices to prove it for R_ℓ . By Lemma 19.2.1, this follows from [44, Th. 4.3 (b)].

The fact that $R^\ell(\text{DM}_{\text{gm}}^{\text{eff}}(k) \otimes \mathbb{Q}) \subset \tilde{D}_m^b(k, \mathbb{Q}_\ell)^{\text{eff}}$ now follows from [SGA7, Exp. XXI, (5.2.2)] which says that the eigenvalues of the geometric Frobenius acting on ℓ -adic cohomology with compact supports are algebraic integers.

The rest of the properties follow from the standard properties of ℓ -adic cohomology, plus the Riemann hypothesis [26]. \square

19.4. Sheaves of level ≤ 1 .

19.4.1. Definition. Let $\mathcal{F} \in \text{Mix}(k, \mathbb{Q}_\ell)$. We say that \mathcal{F} is of level ≤ 1 if

- \mathcal{F} is effective (Def. 19.3.3) and its weights are in $\{0, -1, -2\}$ (Def. 19.3.2);
- $W_{-2}\mathcal{F}(-1)$ is effective.

We write $\text{Mix}(k, \mathbb{Q}_\ell)_{(1)}$ for the full subcategory of $\text{Mix}(k, \mathbb{Q}_\ell)^{\text{eff}}$ consisting of objects of level ≤ 1 , and $\tilde{D}_m^b(k, \mathbb{Q}_\ell)_{(1)}$ the full subcategory of $\tilde{D}_m^b(k, \mathbb{Q}_\ell)$ consisting of objects with cohomology in $\text{Mix}(k, \mathbb{Q}_\ell)_{(1)}$.

Note that this coincides with the definitions in §15.3, with $\text{Mix}(k, \mathbb{Q}_\ell)_{\mathbb{L}} = \{\mathcal{F}(1) \mid \mathcal{F} \text{ effective of weight } 0\}$. In particular, $\tilde{D}_m^b(k, \mathbb{Q}_\ell)_{(1)}$ is a thick triangulated t -subcategory of $\tilde{D}_m^b(k, \mathbb{Q}_\ell)^{\text{eff}}$ with heart $\text{Mix}(k, \mathbb{Q}_\ell)_{(1)}$. Moreover, as a special case of Proposition 15.3.4, we have:

19.4.2. Proposition. *The full embeddings $\iota : \text{Mix}(k, \mathbb{Q}_\ell)_{(1)} \hookrightarrow \text{Mix}(k, \mathbb{Q}_\ell)^{\text{eff}}$ and $\iota : D^b(k, \mathbb{Q}_\ell)_{(1)} \hookrightarrow D^b(k, \mathbb{Q}_\ell)^{\text{eff}}$ have left adjoints Alb^ℓ and LAlb^ℓ .*

\square

19.5. Derived realisation for 1-motives. To get the correct derived realisation for 1-motives, we start from Deligne's l -adic realisation for smooth 1-motives over a base S [25, (10.1.10)]:

$$T_\ell(S) : \mathcal{M}_1(S) \otimes \mathbb{Q} \rightarrow \text{Mix}(S, \mathbb{Q}_\ell).$$

For S running through smooth models of k over \mathbb{F}_p , the $T_\ell(S)$ induce an exact functor

$$T_\ell : \mathcal{M}_1(k) \otimes \mathbb{Q} \rightarrow \text{Mix}(k, \mathbb{Q}_\ell).$$

Deligne's description of $T_\ell(S)$ shows that, in fact, T_ℓ takes its values in $\text{Mix}(k, \mathbb{Q}_\ell)_{(1)}$. From the definition of T_ℓ , we get immediately:

19.5.1. Theorem. *Let T_ℓ still denote the trivial extension of T_ℓ to a t -exact functor $D^b(\mathcal{M}_1(k) \otimes \mathbb{Q}) \rightarrow \tilde{D}_m^b(k, \mathbb{Q}_\ell)$ (via Remark 19.1.4 3)). Then there is a canonical isomorphism $u : T_\ell \xrightarrow{\sim} R_\ell \circ \text{Tot}$. \square*

19.6. An ℓ -adic version of Deligne's conjecture. From the above we get, for any $M \in \text{DM}_{\text{gm}}^{\text{eff}}(k, \mathbb{Q})$, the base change morphism (15.4):

$$(19.6) \quad R_\ell(M)_{\leq 1} \rightarrow T_\ell \text{LAlb}(M).$$

Taking homology of both sides, we get comparison maps

$$(19.7) \quad H_i(R_\ell(M))_{\leq 1} \rightarrow T_\ell \text{LiAlb}(M)$$

for all $i \in \mathbb{Z}$.

Applying Theorem 15.5.5 and the Tate-Zarhin theorem for endomorphisms of abelian varieties, we get:

19.6.1. Theorem.

- (i) *If X is smooth and projective, then (19.7) is an isomorphism for all $i \neq 2$; for $i = 2$, it is an isomorphism if and only if the Tate conjecture holds in codimension 1 for X .*
- (ii) *The map (19.6) is an isomorphism for motives M of abelian type.*
- (iii) *For a general M , (19.6) induces an isomorphism after applying the functor $Q \mapsto \text{cone}(W_{-2}Q \rightarrow Q)$. \square*

Modulo resolution of singularities and the Tate conjecture in codimension 1, we get the same corollaries as in Cor. 17.3.2.

Appendices

APPENDIX A. HOMOLOGICAL ALGEBRA

A.1. Some comparison lemmas. The following lemma is probably well-known:

A.1.1. Lemma. *Let $T : \mathcal{T} \rightarrow \mathcal{T}'$ be a full triangulated functor between two triangulated categories. Then T is conservative if and only if it is faithful.*

Proof. “If” is obvious. For “only if”, let $f : X \rightarrow X'$ be a morphism of \mathcal{T} such that $T(f) = 0$. Let $g : X' \rightarrow X''$ denote a cone of f . Then $T(g)$ has a retraction ρ . Applying fullness, we get an equality $\rho = T(r)$. Applying conservativity, $u = rg$ is an isomorphism. Then $r' = u^{-1}r$ is a retraction of g , which implies that $f = 0$. \square

A.1.2. Proposition. *a) Let $i : \underline{E} \hookrightarrow \underline{A}$ be an exact full sub-category of an abelian category \underline{A} , closed under kernels. Assume further that for each $A^\bullet \in C^b(\underline{A})$ there exists $E^\bullet \in C^b(\underline{E})$ and a quasi-isomorphism $i(E^\bullet) \rightarrow A^\bullet$ in $K^b(\underline{A})$. Then $i : D^b(\underline{E}) \rightarrow D^b(\underline{A})$ is an equivalence of categories.*

b) The hypothesis of a) is granted when every object in \underline{A} has a finite left resolution by objects in \underline{E} .

Proof. a) Clearly, the functor $D^b(\underline{E}) \rightarrow D^b(\underline{A})$ is conservative. The assumption implies that i is essentially surjective: thanks to Lemma A.1.1, in order to conclude it remains to see that i is full.

Let $f \in D^b(\underline{A})(i(D^\bullet), i(E^\bullet))$. Since $D^b(\underline{A})$ has left calculus of fractions there exists a quasi-isomorphism s such that $f = f's^{-1}$ where $f' : A^\bullet \rightarrow i(E^\bullet)$ is a map in $K^b(\underline{A})$, which then lifts to a map in $C^b(\underline{A})$. By hypothesis there exists $F^\bullet \in C^b(\underline{E})$ and a quasi-isomorphism $s' : i(F^\bullet) \rightarrow A^\bullet$. Set $f'' := f's' : i(F^\bullet) \rightarrow i(E^\bullet)$. Then $f = f''(ss')^{-1}$ where $ss' : i(F^\bullet) \rightarrow i(D^\bullet)$ is a quasi-isomorphism. By conservativity of i , we are reduced to check fullness for effective maps, *i.e.* arising from true maps in $C^b(\underline{A})$: this easily follows from the fullness of the functor $C^b(\underline{E}) \hookrightarrow C^b(\underline{A})$.

b) This follows by adapting the argument in [39, I, Lemma 4.6]. \square

A.1.3. Proposition. *Let $\underline{E} \hookrightarrow \underline{A}$ be an exact category. Let D be a triangulated category and let $T : D^b(\underline{E}) \rightarrow D$ be a triangulated functor such that*

$$\mathrm{Hom}_{D^b(\underline{E})}(E', E[i]) \xrightarrow{\cong} \mathrm{Hom}_D(T(E'), T(E[i]))$$

for all $E', E \in \underline{E}$ and $i \in \mathbb{Z}$. Then T is fully faithful.

Proof. Let $C, C' \in C^b(\underline{E})$: we want to show that the map

$$\mathrm{Hom}_{D^b(\underline{E})}(C, C') \rightarrow \mathrm{Hom}_D(T(C), T(C'))$$

is bijective. We argue by induction on the lengths of C and C' . \square

Finally, we have the following very useful criterion for a full embedding of derived categories, that we learned from Pierre Schapira.

A.1.4. Proposition ([51, p. 329, Th. 13.2.8]). *Let $\mathcal{A} \hookrightarrow \mathcal{B}$ be an exact full embedding of abelian categories. Assume that, given any monomorphism $X' \hookrightarrow X$ in \mathcal{B} , with $X' \in \mathcal{A}$, there exists a morphism $X \rightarrow X''$, with $X'' \in \mathcal{A}$, such that the composite morphism $X \rightarrow X''$ is a monomorphism. Then the functor*

$$D^*(\mathcal{A}) \rightarrow D^*(\mathcal{B})$$

is fully faithful for $$ = +, b.*

A.2. The Tot construction.

A.2.1. Lemma. *Let \mathcal{A} be an abelian category and let $\mathcal{A}^{[0,1]}$ be the (abelian) category of complexes of length 1 of objects of \mathcal{A} . Then the “total complex” functor induces a triangulated functor*

$$D^*(\mathcal{A}^{[0,1]}) \rightarrow D^*(\mathcal{A})$$

for any decoration $$.*

Proof. We may consider a complex of objects of $\mathcal{A}^{[0,1]}$ as a double complex of objects of \mathcal{A} and take the associated total complex. This yields a functor

$$\text{Tot} : C^*(\mathcal{A}^{[0,1]}) \rightarrow D^*(\mathcal{A}).$$

(Note that if we consider a complex of objects of $\mathcal{A}^{[0,1]}$

$$M^\bullet = [L^\bullet \xrightarrow{u^\bullet} G^\bullet]$$

as a map $u^\bullet : L^\bullet \rightarrow G^\bullet$ of complexes of \mathcal{A} , then $\text{Tot}(M^\bullet)$ coincides with the cone of u^\bullet .)

This functor factors through a triangulated functor from $D^*(\mathcal{A}^{[0,1]})$: indeed it is easily checked that a) Tot preserves homotopies, b) the induced functor on $K^*(\mathcal{A}^{[0,1]})$ is triangulated; c) Tot of an acyclic complex is 0 (which follows from a spectral sequence argument). \square

A.2.2. Lemma. *With notation as in Lemma A.2.1, the set of q.i. of $\mathcal{A}^{[0,1]}$ enjoys a calculus of left and right fractions within the homotopy category $K(\mathcal{A}^{[0,1]})$ (same objects, morphisms modulo the homotopy relation).*

Proof. It is enough to show the calculus of right fractions (for left fractions, replace \mathcal{A} by \mathcal{A}^{op}).

a) Let

$$\begin{array}{ccc} & \tilde{C} & \\ & \downarrow u & \\ D & \xrightarrow{f} & C \end{array}$$

be a diagram in $\mathcal{A}^{[0,1]}$, with u a q.i.. Consider the mapping fibre (= shifted mapping cone) F of the map $\tilde{C} \oplus D \rightarrow C$. The complex F is in general concentrated in degrees $[0, 1, 2]$; however, since $F \rightarrow D$ is a q.i., the truncation $\tau_{<2}F$ is q.i. to F ; then $\tilde{D} = \tau_{<2}F$ fills in the square in $K(\mathcal{A}^{[0,1]})$.

b) Let

$$[C^0 \rightarrow C^1] \xrightarrow{f} [D^0 \rightarrow D^1] \xrightarrow{u} [\tilde{D}^0 \rightarrow \tilde{D}^1]$$

be a chain of maps of $\mathcal{A}^{[0,1]}$ such that u is a q.i. and uf is homotopic to 0. Let $s : C^1 \rightarrow \tilde{D}^0$ be a corresponding homotopy. Define \tilde{C}^1 as the fibre product of C^1 and D^0 over \tilde{D}^0 (via s and u^0), $\tilde{s} : \tilde{C}^1 \rightarrow D^0$ the corresponding map and \tilde{C}^0 the fibre product of C^0 and \tilde{C}^1 over C^1 . One then checks that $v : [\tilde{C}^0 \rightarrow \tilde{C}^1] \rightarrow [C^0 \rightarrow C^1]$ is a q.i. and that \tilde{s} defines a homotopy from fv to 0. \square

A.2.3. Remark. One can probably extend these two lemmas to complexes of a fixed length n by the same arguments: we leave this to the interested reader.

APPENDIX B. TORSION OBJECTS IN ADDITIVE CATEGORIES

B.1. Additive categories.

B.1.1. Definition. Let \mathcal{A} be an additive category, and let A be a subring of \mathbb{Q} .

a) We write $\mathcal{A} \otimes A$ for the category with the same objects as \mathcal{A} but morphisms

$$(\mathcal{A} \otimes A)(X, Y) := \mathcal{A}(X, Y) \otimes A.$$

b) We denote by $\mathcal{A}\{A\}$ the full subcategory of \mathcal{A} :

$$\{X \in \mathcal{A} \mid \exists n > 0 \text{ invertible in } A, n1_X = 0\}.$$

For $A = \mathbb{Q}$, we write $\mathcal{A}\{A\} = \mathcal{A}_{\text{tors}}$. We say that $X \in \mathcal{A}\{A\}$ is an *A-torsion object* (a torsion object if $A = \mathbb{Q}$).

c) A morphism $f : X \rightarrow Y$ in \mathcal{A} is an *A-isogeny* (an *isogeny* if $A = \mathbb{Q}$) if there exists a morphism $g : Y \rightarrow X$ and an integer n invertible in A such that $fg = n1_Y$ and $gf = n1_X$. We denote by $\Sigma_A(\mathcal{A})$ the collection of *A-isogenies* of \mathcal{A} .

d) We say that two objects $X, Y \in \mathcal{A}$ are *A-isogenous* if they can be

linked by a chain of A -isogenies (not necessarily pointing in the same direction).

B.1.2. Proposition. *a) The subcategory $\mathcal{A}\{A\}$ is additive and closed under direct summands.*

b) The A -isogenies $\Sigma_A(\mathcal{A})$ form a multiplicative system of morphisms in \mathcal{A} , enjoying calculi of left and right fractions. The corresponding localisation of \mathcal{A} is isomorphic to $\mathcal{A} \otimes A$.

Proof. a) is clear. For b), consider the obvious functor $P : \mathcal{A} \rightarrow \mathcal{A} \otimes A$. We claim that

$$\Sigma_A(\mathcal{A}) = \{f \mid P(f) \text{ is invertible.}\}$$

One inclusion is clear. Conversely, let $f : X \rightarrow Y$ be such that $P(f)$ is invertible. This means that there exists $\gamma \in (\mathcal{A} \otimes A)(Y, X)$ such that $P(f)\gamma = 1_Y$ and $\gamma P(f) = 1_X$. Choose an integer $m \in A - \{0\}$ such that $m\gamma = P(g_1)$ for some g_1 . Then there is another integer $n \in A - \{0\}$ such that

$$n(fg_1 - m1_Y) = 0 \text{ and } n(g_1f - m1_X) = 0.$$

Taking $g = ng_1$ shows that $f \in \Sigma_A(\mathcal{A})$.

It is also clear that homotheties by nonzero integers of A form a cofinal system in $\Sigma_A(\mathcal{A})$. This shows immediately that we have calculi of left and right fractions.

It remains to show that the induced functor

$$\Sigma_A(\mathcal{A})^{-1}\mathcal{A} \rightarrow \mathcal{A} \otimes A$$

is an isomorphism of categories; but this is immediate from the well-known formula, in the presence of calculus of fractions:

$$\Sigma_A(\mathcal{A})^{-1}\mathcal{A}(X, Y) = \varinjlim_{X' \xrightarrow{f} X \in \Sigma} \mathcal{A}(X', Y) = \varinjlim_{X \xrightarrow{n} X, n \in A - \{0\}} \mathcal{A}(X, Y).$$

□

The following lemma is clear.

B.1.3. Lemma. *Let \mathcal{B} be a full additive subcategory of \mathcal{A} , and suppose that every object of \mathcal{B} is A -isogenous to an object of \mathcal{A} . Then $\mathcal{B} \otimes A \xrightarrow{\sim} \mathcal{A} \otimes A$.* □

B.2. Triangulated categories. (See [73, A.2.1] for a different treatment.)

B.2.1. Proposition. *Let \mathcal{T} be a triangulated category. Then*

a) The subcategory $\mathcal{T}\{A\}$ is triangulated and thick.

b) Let $X \in \mathcal{T}$ and $n \in A - \{0\}$. Then “the” cone X/n of multiplication

by n on X belongs to $\mathcal{T}\{A\}$.

c) The localised category $\mathcal{T}/\mathcal{T}\{A\}$ is canonically isomorphic to $\mathcal{T} \otimes A$. In particular, $\mathcal{T} \otimes A$ is triangulated.

d) A morphism f of \mathcal{T} belongs to $\Sigma_A(\mathcal{T})$ if and only if $\text{cone}(f) \in \mathcal{T}\{A\}$.

Proof. a) It is clear that $\mathcal{T}\{A\}$ is stable under direct summands; it remains to see that it is triangulated. Let $X, Y \in \mathcal{T}\{A\}$, $f : X \rightarrow Y$ a morphism and Z a cone of f . We may assume that $n1_X = n1_Y = 0$. The commutative diagram

$$\begin{array}{ccccc} Y & \longrightarrow & Z & \longrightarrow & X[1] \\ n=0 \downarrow & & n \downarrow & & n=0 \downarrow \\ Y & \longrightarrow & Z & \longrightarrow & X[1] \end{array}$$

show that multiplication by n on Z factors through Y ; this implies that $n^2 1_Z = 0$.

b) Exactly the same argument as in a) shows that multiplication by n on X/n factors through X , hence that $n^2 1_{X/n} = 0$.

c) Let $f \in \mathcal{T}$ be such that $C := \text{cone}(f) \in \mathcal{T}\{A\}$, and let $n > 0$ be such that $n1_C = 0$. The same trick as in a) and b) shows that there exist factorisations $n = ff' = f''f$, hence that $f \in \Sigma_A(\mathcal{T})$. In particular, f becomes invertible under the canonical (additive) functor $\mathcal{T} \rightarrow \mathcal{T} \otimes A$. Hence an induced (additive) functor

$$\mathcal{T}/\mathcal{T}\{A\} \rightarrow \mathcal{T} \otimes A$$

which is evidently bijective on objects; b) shows immediately that it is fully faithful.

d) One implication has been seen in the proof of c). For the other, if $f \in \Sigma_A(\mathcal{T})$, then f becomes invertible in $\mathcal{A} \otimes A$, hence $\text{cone}(f) \in \mathcal{T}\{A\}$ by c). \square

B.2.2. Remark. As is well-known, the stable homotopy category gives a counterexample to the expectation that in fact $n1_{X/n} = 0$ in b) ($X = S^0$, $n = 2$).

We now show that $\otimes A$ is a “flat” operation on triangulated categories.

B.2.3. Lemma. Let $0 \rightarrow \mathcal{T}' \rightarrow \mathcal{T} \rightarrow \mathcal{T}'' \rightarrow 0$ be a short exact sequence of triangulated categories (by definition, this means that \mathcal{T}' is strictly full in \mathcal{T} , stable under cones and shifts, and that \mathcal{T}'' is equivalent to \mathcal{T}/\mathcal{T}'). Then the sequence

$$0 \rightarrow \mathcal{T}' \otimes A \rightarrow \mathcal{T} \otimes A \rightarrow \mathcal{T}'' \otimes A \rightarrow 0$$

is exact.

Proof. By Proposition B.2.1 b), all categories remain triangulated after $\otimes A$ and the induced functors are clearly triangulated functors. Since $\mathcal{T}' \rightarrow \mathcal{T}$ is strictly full, so is $\mathcal{T}' \otimes A \rightarrow \mathcal{T} \otimes A$. If f is a morphism in $\mathcal{T}' \otimes A$, then f is the composition of an isomorphism with a morphism coming from \mathcal{T}' , whose cone in \mathcal{T} lies in \mathcal{T}' , thus the cone of f in $\mathcal{T} \otimes A$ lies in $\mathcal{T}' \otimes A$.

We now show that the functor

$$a : \frac{\mathcal{T} \otimes A}{\mathcal{T}' \otimes A} \rightarrow \mathcal{T}'' \otimes A$$

is an equivalence of categories. Since the left hand side is A -linear, the natural functor

$$\mathcal{T}/\mathcal{T}' \rightarrow \mathcal{T} \otimes A/\mathcal{T}' \otimes A$$

canonically extends to a functor

$$b : (\mathcal{T}/\mathcal{T}') \otimes A \rightarrow \mathcal{T} \otimes A/\mathcal{T}' \otimes A.$$

It is clear that a and b are inverse to each other. \square

B.2.4. Proposition. *a) Let $T : \mathcal{S} \rightarrow \mathcal{T}$ be a triangulated functor between triangulated categories. Then T is fully faithful if and only if the induced functors $T\{A\} : \mathcal{S}\{A\} \rightarrow \mathcal{T}\{A\}$ and $T \otimes A : \mathcal{S} \otimes A \rightarrow \mathcal{T} \otimes A$ are fully faithful.*

b) Assuming a) holds, T is an equivalence of categories if and only if $T\{A\}$ and $T \otimes A$ are.

Proof. a) “Only if” is obvious; let us prove “if”. Let $X, Y \in \mathcal{S}$: we have to prove that $T : \mathcal{S}(X, Y) \rightarrow \mathcal{T}(T(X), T(Y))$ is bijective. We do it in two steps:

1) Y is torsion, say $n1_Y = 0$. The claim follows from the commutative diagram with exact rows

$$\begin{array}{ccccccc} \mathcal{S}(X/n[1], Y) & \rightarrow & \mathcal{S}(X, Y) & \xrightarrow{n=0} & \mathcal{S}(X, Y) & \rightarrow & \mathcal{S}(X/n, Y) \\ T \downarrow \wr & & T \downarrow & & T \downarrow & & T \downarrow \wr \\ \mathcal{T}(T(X)/n[1], T(Y)) & \rightarrow & \mathcal{T}(T(X), T(Y)) & \xrightarrow{n=0} & \mathcal{T}(T(X), T(Y)) & \rightarrow & \mathcal{T}(T(X)/n, T(Y)) \end{array}$$

and the assumption (see Proposition B.2.1 b)).

2) The general case. Let $n > 0$. We have a commutative diagram with exact rows

$$\begin{array}{ccccccc} 0 \rightarrow & \mathcal{S}(X, Y)/n & \rightarrow & \mathcal{S}(X, Y/n) & \rightarrow & {}_n\mathcal{S}(X, Y[1]) & \rightarrow 0 \\ & T \downarrow & & T \downarrow \wr & & T \downarrow & \\ 0 \rightarrow & \mathcal{T}(T(X), T(Y))/n & \rightarrow & \mathcal{T}(T(X), T(Y)/n) & \rightarrow & {}_n\mathcal{T}(T(X), T(Y)[1]) & \rightarrow 0 \end{array}$$

where the middle isomorphism follows from 1). The snake lemma yields an exact sequence

$$\begin{aligned} 0 \rightarrow \mathcal{S}(X, Y)/n \xrightarrow{T} \mathcal{T}(T(X), T(Y))/n \\ \rightarrow {}_n\mathcal{S}(T(X), T(Y)[1]) \xrightarrow{T} {}_n\mathcal{T}(T(X), T(Y)[1]) \rightarrow 0. \end{aligned}$$

Passing to the limit over n , we get another exact sequence

$$\begin{aligned} \text{(B.1)} \quad 0 \rightarrow \mathcal{S}(X, Y) \otimes A/\mathbb{Z} \xrightarrow{T} \mathcal{T}(T(X), T(Y)) \otimes A/\mathbb{Z} \\ \rightarrow \mathcal{S}(T(X), T(Y)[1])\{A\} \xrightarrow{T} \mathcal{T}(T(X), T(Y)[1])\{A\} \rightarrow 0. \end{aligned}$$

Consider now the commutative diagram with exact rows

$$\begin{array}{ccccccc} 0 \rightarrow & \mathcal{S}(X, Y)\{A\} & \rightarrow & \mathcal{S}(X, Y) & \rightarrow & \mathcal{S}(X, Y) \otimes A & \rightarrow \mathcal{S}(X, Y) \otimes A/\mathbb{Z} \rightarrow 0 \\ & T \downarrow \underline{1} & & T \downarrow \underline{2} & & T \downarrow \wr & & T \downarrow \underline{4} \\ 0 \rightarrow & \mathcal{T}(T(X), T(Y))\{A\} & \rightarrow & \mathcal{T}(T(X), T(Y)) & \rightarrow & \mathcal{T}(T(X), T(Y)) \otimes A & \rightarrow \mathcal{T}(T(X), T(Y)) \otimes A/\mathbb{Z} \rightarrow 0 \end{array}$$

where the isomorphism is by assumption. By this diagram and (B.1), $\underline{4}$ is an isomorphism. Using this fact in (B.1) applied with $Y[-1]$, we get that $\underline{1}$ is an isomorphism; then $\underline{2}$ is an isomorphism by the 5 lemma, as desired.

b) If T is essentially surjective, so is $T \otimes A$, as well as $T\{A\}$ as long as T is faithful, which is implied by a). Conversely, let $X \in \mathcal{T}$. Using only the essential surjectivity of $T \otimes A$, we find an A -isogeny

$$\varphi : X \rightarrow T(Y)$$

with $Y \in \mathcal{S}$. A cone of φ is a torsion object, hence, if $T\{A\}$ is essentially surjective, it is isomorphic to $T(C)$ for $C \in \mathcal{S}\{A\}$. Thus X sits in an exact triangle

$$X \xrightarrow{\varphi} T(Y) \xrightarrow{\psi} T(C) \xrightarrow{+1}.$$

If T is full, then $\psi = T(\psi')$ and $X \simeq T(X')$, where X' is a fibre of ψ' . \square

B.3. Torsion objects in an abelian category. The proof of the following proposition is similar to that of Proposition B.2.1 and is left to the reader.

B.3.1. Proposition. *Let \mathcal{A} be an abelian category. Then*

- a) *The full subcategory $\mathcal{A}\{A\}$ is thick (a Serre subcategory, in another terminology).*
- b) *Let $X \in \mathcal{A}$ and $n > 0$ invertible in A . Then the kernel and cokernel of multiplication by n on X belong to $\mathcal{A}\{A\}$.*
- c) *The localised category $\mathcal{A}/\mathcal{A}\{A\}$ is canonically isomorphic to $\mathcal{A} \otimes A$.*

In particular, $\mathcal{A} \otimes A$ is abelian.

d) A morphism $f \in \mathcal{A}$ is in $\Sigma_A(\mathcal{A})$ if and only if $\text{Ker } f \in \mathcal{A}\{A\}$ and $\text{Coker } f \in \mathcal{A}\{A\}$. \square

The following corollary is a direct consequence of Proposition B.3.1 and Lemma B.1.3:

B.3.2. Corollary. *Let \mathcal{A} be an abelian category. Let \mathcal{B} be a full additive subcategory of \mathcal{A} , and suppose that every object of \mathcal{B} is A -isogenous to an object of \mathcal{A} (see Definition B.1.1). Then $\mathcal{B} \otimes A$ is abelian, and in particular idempotent-complete.* \square

B.4. Abelian and derived categories.

B.4.1. Proposition. *Let \mathcal{A} be an abelian category. Then the natural functor $D^b(\mathcal{A}) \rightarrow D^b(\mathcal{A} \otimes A)$ induces an equivalence of categories*

$$D^b(\mathcal{A}) \otimes A \xrightarrow{\sim} D^b(\mathcal{A} \otimes A).$$

In particular, $D^b(\mathcal{A}) \otimes A$ is idempotent-complete.

Proof. In 3 steps:

1) The natural functor $C^b(\mathcal{A}) \otimes A \rightarrow C^b(\mathcal{A} \otimes A)$ is an equivalence of categories. Full faithfulness is clear. For essential surjectivity, take a bounded complex C of objects of $\mathcal{A} \otimes A$. Find a common denominator to all differentials involved in C . Then the corresponding morphisms of \mathcal{A} have torsion composition; since they are finitely many, we may multiply by a common bigger integer so that they compose to 0. The resulting complex of $C^b(\mathcal{A})$ then becomes isomorphic to C in $C^b(\mathcal{A} \otimes A)$.

2) The functor of 1) induces an equivalence of categories $K^b(\mathcal{A}) \otimes A \xrightarrow{\sim} K^b(\mathcal{A} \otimes A)$. Fullness is clear, and faithfulness is obtained by the same technique as in 1).

3) The functor of 2) induces the desired equivalence of categories. First, the functor

$$D^b(\mathcal{A})/D_{\mathcal{A}\{A\}}^b(\mathcal{A}) \rightarrow D^b(\mathcal{A}/\mathcal{A}\{A\})$$

is obviously conservative. But clearly $D_{\mathcal{A}\{A\}}^b(\mathcal{A}) = D^b(\mathcal{A})\{A\}$. Hence, by Propositions B.2.1 and B.3.1, this functor translates as

$$D^b(\mathcal{A}) \otimes A \rightarrow D^b(\mathcal{A} \otimes A).$$

Let $A^b(\mathcal{A})$ denote the thick subcategory of $K^b(\mathcal{A})$ consisting of acyclic complexes. By Lemma B.2.3 we have a commutative diagram of exact

sequences of triangulated categories

$$\begin{array}{ccccccc}
0 & \longrightarrow & A^b(\mathcal{A}) \otimes A & \longrightarrow & K^b(\mathcal{A}) \otimes A & \longrightarrow & D^b(\mathcal{A}) \otimes A \longrightarrow 0 \\
& & \downarrow & & \downarrow & & \downarrow \\
0 & \longrightarrow & A^b(\mathcal{A} \otimes A) & \longrightarrow & K^b(\mathcal{A} \otimes A) & \longrightarrow & D^b(\mathcal{A} \otimes A) \longrightarrow 0.
\end{array}$$

We have just seen that the right vertical functor is conservative, and by 2), the middle one is an equivalence. Hence the left one is essentially surjective. By the same argument as in the proof of Proposition B.2.4 b), we get that the right functor is full, and the result follows from Lemma A.1.1. \square

APPENDIX C. 1-MOTIVES WITH TORSION

Effective 1-motives which admit torsion are introduced in [7, §1] (in characteristic 0). We investigate some properties (over a perfect field of exponential characteristic $p \geq 1$) which are not included in op. cit. as a supplement to our Sect. 1.

C.1. Effective 1-motives. An *effective 1-motive with torsion* over k is a complex of group schemes $M = [L \xrightarrow{u} G]$ where L is finitely generated locally constant for the étale topology *i.e.* a discrete sheaf of Def. 1.1.1, and G is a semi-abelian k -scheme. Therefore L can be represented by an extension

$$0 \rightarrow L_{\text{tor}} \rightarrow L \rightarrow L_{\text{fr}} \rightarrow 0$$

where L_{tor} is a finite étale k -group scheme and L_{fr} is free, *i.e.* a lattice. Also G can be represented by an extension of an abelian k -scheme A by a k -torus T .

C.1.1. Definition. An *effective* map from $M = [L \xrightarrow{u} G]$ to $M' = [L' \xrightarrow{u'} G']$ is a commutative square

$$\begin{array}{ccc}
L & \xrightarrow{u} & G \\
f \downarrow & & \downarrow g \\
L' & \xrightarrow{u'} & G'
\end{array}$$

in the category of group schemes. Denote by $(f, g) : M \rightarrow M'$ such a map. The natural composition of squares makes up a category, denoted by ${}^t\mathcal{M}_1^{\text{eff}}$. We will denote by $\text{Hom}_{\text{eff}}(M, M')$ the abelian group of effective morphisms.

For a given 1-motive $M = [L \xrightarrow{u} G]$ we have (in the abelian category of commutative group schemes) a commutative diagram

$$\begin{array}{ccccccc}
 & & 0 & & 0 & & \\
 & & \downarrow & & \downarrow & & \\
 0 & \longrightarrow & \text{Ker}(u) \cap L_{\text{tor}} & \longrightarrow & L_{\text{tor}} & \xrightarrow{u} & u(L_{\text{tor}}) \longrightarrow 0 \\
 & & \downarrow & & \downarrow & & \downarrow \\
 \text{(C.1)} \quad 0 & \longrightarrow & \text{Ker}(u) & \longrightarrow & L & \xrightarrow{u} & G \\
 & & & & \downarrow & & \downarrow \\
 & & & & L_{\text{fr}} & \xrightarrow{\bar{u}} & G/u(L_{\text{tor}}) \\
 & & & & \downarrow & & \downarrow \\
 & & & & 0 & & 0
 \end{array}$$

with exact rows and columns. We set

- $M_{\text{fr}} := [L_{\text{fr}} \xrightarrow{\bar{u}} G/u(L_{\text{tor}})]$
- $M_{\text{tor}} := [\text{Ker}(u) \cap L_{\text{tor}} \rightarrow 0]$
- $M_{\text{tf}} := [L/\text{Ker}(u) \cap L_{\text{tor}} \xrightarrow{u} G]$

considered as effective 1-motives. From Diagram (C.1) there are canonical effective maps $M \rightarrow M_{\text{tf}}$, $M_{\text{tor}} \rightarrow M$ and $M_{\text{tf}} \rightarrow M_{\text{fr}}$.

C.1.2. Definition. A 1-motive $M = [L \xrightarrow{u} G]$ is *free* if L is free, *i.e.* if $M = M_{\text{fr}}$. M is *torsion* if L is torsion and $G = 0$, *i.e.* if $M = M_{\text{tor}}$, and *torsion-free* if $\text{Ker}(u) \cap L_{\text{tor}} = 0$, *i.e.* if $M = M_{\text{tf}}$.

Denote by ${}^t\mathcal{M}_1^{\text{eff,fr}}$, ${}^t\mathcal{M}_1^{\text{eff,tor}}$ and ${}^t\mathcal{M}_1^{\text{eff,tf}}$, the full sub-categories of ${}^t\mathcal{M}_1^{\text{eff}}$ given by free, torsion and torsion-free 1-motives respectively.

The category ${}^t\mathcal{M}_1^{\text{eff,fr}}$ is nothing else than the category \mathcal{M}_1 of Deligne 1-motives and we shall henceforth use this notation. It is clear that ${}^t\mathcal{M}_1^{\text{eff,tor}}$ is equivalent to the category of finite étale group schemes. If M is torsion-free then Diagram (C.1) is a pull-back, *i.e.* L is the pull-back of L_{fr} along the isogeny $G \rightarrow G/L_{\text{tor}}$.

C.1.3. Proposition. *The categories ${}^t\mathcal{M}_1^{\text{eff}}$ and \mathcal{M}_1 have all finite limits and colimits.*

Proof. Since these are additive categories (with biproducts), it is enough to show that they have kernels, dually cokernels. Now let $\varphi = (f, g) : M \rightarrow M'$ be an effective map. We claim that

$$\text{Ker } \varphi = [\text{Ker}^0(f) \xrightarrow{u} \text{Ker}^0(g)]$$

is the pull-back of $\mathrm{Ker}^0(g)$ along u , where $\mathrm{Ker}^0(g)$ is the (reduced) connected component of the identity of the kernel of $g : G \rightarrow G'$ and $\mathrm{Ker}^0(f) \subseteq \mathrm{Ker}(f)$. We have to show that the following diagram of effective 1-motives

$$\begin{array}{ccc}
 0 & & 0 \\
 \downarrow & & \downarrow \\
 \mathrm{Ker}^0(f) & \xrightarrow{u} & \mathrm{Ker}^0(g) \\
 \downarrow & & \downarrow \\
 L & \xrightarrow{u} & G \\
 f \downarrow & & g \downarrow \\
 L' & \xrightarrow{u'} & G'
 \end{array}$$

satisfies the universal property for kernels. Suppose that $M'' = [L'' \xrightarrow{u''} G'']$ is mapping to M in such a way that the composition $M'' \rightarrow M \rightarrow M'$ is the zero map. Then L'' maps to $\mathrm{Ker}(f)$ and G'' maps to $\mathrm{Ker}(g)$. Since G'' is connected, it actually maps to $\mathrm{Ker}^0(g)$ and, by the universal property of pull-backs in the category of group schemes, L'' then maps to $\mathrm{Ker}^0(f)$. Finally note that if L is free then also $\mathrm{Ker}^0(f)$ is free.

For cokernels, we see that

$$[\mathrm{Coker}(f) \xrightarrow{\bar{u}'} \mathrm{Coker}(g)]$$

is an effective 1-motive which is clearly a cokernel of φ .

For \mathcal{M}_1 , it is enough to take the free part of the cokernel, *i.e.* given $(f, g) : M \rightarrow M'$ then $[\mathrm{Coker}(f) \rightarrow \mathrm{Coker}(g)]_{\mathrm{fr}}$ meets the universal property for coker of free 1-motives. \square

C.2. Quasi-isomorphisms. (*cf.* [7, §1]).

C.2.1. Definition. An effective morphism of 1-motives $M \rightarrow M'$, here $M = [L \xrightarrow{u} G]$ and $M' = [L' \xrightarrow{u'} G']$, is a *quasi-isomorphism* (q.i. for

short) of 1-motives if it yields a pull-back diagram

$$(C.2) \quad \begin{array}{ccc} 0 & & 0 \\ \downarrow & & \downarrow \\ F & \xlongequal{\quad} & F \\ \downarrow & & \downarrow \\ L & \xrightarrow{u} & G \\ \downarrow & & \downarrow \\ L' & \xrightarrow{u'} & G' \\ \downarrow & & \downarrow \\ 0 & & 0 \end{array}$$

where F is a finite étale group.

C.2.2. Remarks. 1) Note that kernel and cokernel of a quasi-isomorphism of 1-motives are 0 but, in general, a quasi-isomorphism is not an isomorphism in ${}^t\mathcal{M}_1^{\text{eff}}$. Hence the category ${}^t\mathcal{M}_1^{\text{eff}}$ is not abelian.

2) A q.i. of 1-motives $M \rightarrow M'$ is actually a q.i. of complexes of group schemes. In fact, an effective map of 1-motives $M \rightarrow M'$ is a q.i. of complexes if and only if we have the following diagram

$$\begin{array}{ccccccc} 0 \rightarrow \text{Ker}(u) & \longrightarrow & L & \xrightarrow{u} & G & \longrightarrow & \text{Coker}(u) \rightarrow 0 \\ & & \downarrow & & \downarrow & & \parallel \\ 0 \rightarrow \text{Ker}(u') & \longrightarrow & L' & \xrightarrow{u'} & G' & \longrightarrow & \text{Coker}(u') \rightarrow 0. \end{array}$$

Therefore Ker and Coker of $L \rightarrow L'$ and $G \rightarrow G'$ are equal. Then $\text{Coker}(G \rightarrow G') = 0$, since it is connected and discrete, and $\text{Ker}(G \rightarrow G')$ is a finite group. Conversely, Diagram (C.2) clearly yields a q.i. of complexes. In particular, it easily follows that the class of q.i. of 1-motives is closed under composition of effective morphisms.

C.2.3. Proposition. *Quasi-isomorphisms are simplifiable on the left and on the right.*

Proof. The assertion “on the right” is obvious since the two components of a q.i. are epimorphisms. For the left, let $\varphi = (f, g) : M \rightarrow M'$ and

$\sigma = (s, t) : M' \rightarrow \widetilde{M}$ a q.i. such that $\sigma\varphi = 0$. In the diagram

$$\begin{array}{ccc} L & \xrightarrow{u} & G \\ f \downarrow & & \downarrow g \\ L' & \xrightarrow{u'} & G' \\ s \downarrow & & \downarrow s \\ \widetilde{L} & \xrightarrow{\widetilde{u}} & \widetilde{G} \end{array}$$

we have $\widetilde{L} = L'/F$, $\widetilde{G} = G'/F$, for some finite group F , $\text{Im}(f) \subseteq F$ and $\text{Im}(g) \subseteq F$. Now u' restricts to the identity on F thus $\text{Im}(f) \subseteq \text{Im}(g)$ and $\text{Im}(g) = 0$, since $\text{Im}(g)$ is connected, hence $\varphi = 0$. \square

C.2.4. Proposition. *The class of q.i. admits a calculus of right fractions in the sense of (the dual of) [35, Ch. I, §2.3].*

Proof. By [7, Lemma 1.2], the first condition of calculus of right fractions is verified, and Proposition C.2.3 shows that the second one is verified as well. (Note that we only consider isogenies with étale kernel here.) \square

C.2.5. Remark. The example of the diagram

$$\begin{array}{ccc} [L \rightarrow G] & \xrightarrow{\sigma} & [L' \rightarrow G'] \\ (1,0) \downarrow & & \\ [L \rightarrow 0] & & \end{array}$$

where σ is a nontrivial q.i. shows that calculus of left fractions fails in general.

C.2.6. Lemma. *Let s, t, u be three maps in ${}^t\mathcal{M}_1^{\text{eff}}$, with $su = t$. If s and t are q.i., then so is u .* \square

Proof. Consider the exact sequence of complexes of sheaves

$$0 \rightarrow \text{Ker } u \rightarrow \text{Ker } t \rightarrow \text{Ker } s \rightarrow \text{Coker } u \rightarrow \text{Coker } t \rightarrow \text{Coker } s \rightarrow 0.$$

Since s and t are q.i., the last two terms are 0. Hence $\text{Coker}(u) = [L \rightarrow G]$ is a quotient of $\text{Ker}(s)$; since G is connected, we must have $G = 0$. On the other hand, as a cokernel of a map of acyclic complexes of length 1, $\text{Coker}(u)$ is acyclic, hence $L = 0$. Similarly, $\text{Ker}(u)$ is acyclic. \square

C.3. 1-motives. We now define the category of 1-motives with torsion from ${}^t\mathcal{M}_1^{\text{eff}}$ by formally inverting quasi-isomorphisms.

C.3.1. Definition. The category ${}^t\mathcal{M}_1$ of 1-motives with torsion is the localisation of ${}^t\mathcal{M}_1^{\text{eff}}$ with respect to the multiplicative class $\{\text{q.i.}\}$ of quasi-isomorphisms.

C.3.2. Remark. Note that there are no nontrivial q.i. between free (or torsion) 1-motives. However, the canonical map $M_{\text{tf}} \rightarrow M_{\text{fr}}$ is a quasi-isomorphism (it is an effective isomorphism when $u(L_{\text{tor}}) = 0$).

It follows from Proposition C.2.4 and [35, Ch. I, Prop. 2-4] that the Hom sets in ${}^t\mathcal{M}_1$ are given by the formula

$$\text{Hom}(M, M') = \varinjlim_{\text{q.i.}} \text{Hom}_{\text{eff}}(\widetilde{M}, M')$$

where the limit is taken over the filtering set of all quasi-isomorphisms $\widetilde{M} \rightarrow M$. A morphism of 1-motives $M \rightarrow M'$ can be represented as a diagram

$$\begin{array}{ccc} M & & M' \\ \text{q.i.} \swarrow & & \nearrow \text{eff} \\ & \widetilde{M} & \end{array}$$

and the composition is given by the existence of a \widehat{M} making the following diagram

$$\begin{array}{ccccc} M & & M' & & M'' \\ \text{q.i.} \swarrow & & \nearrow \swarrow & & \nearrow \text{eff} \\ & \widetilde{M} & & \widetilde{M}' & \\ & \text{q.i.} \swarrow & & \nearrow \text{eff} & \\ & & \widehat{M} & & \end{array}$$

commutative. (This \widehat{M} is in fact unique, see [7, Lemma 1.2].)

C.4. Strict morphisms. The notion of strict morphism is essential in order to show that the $\mathbb{Z}[1/p]$ -linear category of 1-motives with torsion is abelian (*cf.* [7, §1]).

C.4.1. Definition. We say that an effective morphism $(f, g) : M \rightarrow M'$ is *strict* if we have

$$\text{Ker}(f, g) = [\text{Ker}(f) \rightarrow \text{Ker}(g)]$$

i.e. if $\text{Ker}(g)$ is (connected) semiabelian.

To get a feeling on the notion of strict morphism, note:

C.4.2. Lemma. *Let $\varphi = (f, g) : M \rightarrow N$ be a strict morphism, with g onto. Suppose that $\varphi = \sigma \tilde{\varphi}$, where σ is a q.i. Then σ is an isomorphism.*

Conversely, we obtain:

C.4.3. Proposition ([7, Prop. 1.3]). *Any effective morphism $\varphi \in \text{Hom}_{\text{eff}}(M, M')$ can be factored as follows*

$$(C.3) \quad \begin{array}{ccc} M & \xrightarrow{\varphi} & M' \\ \tilde{\varphi} \searrow & & \nearrow \\ & \widetilde{M} & \end{array}$$

where $\tilde{\varphi}$ is a strict morphism and $\widetilde{M} \rightarrow M'$ is a q.i. or a p -power isogeny.

Proof. (Sketch) Note that if $\varphi = (f, g)$ we always have the following natural factorisation of the map g between semi-abelian schemes

$$\begin{array}{ccc} G & \xrightarrow{g} & G' \\ \searrow & & \nearrow \\ G/\text{Ker}^0(g) & & \end{array}$$

If g is a surjection we get the claimed factorisation by taking $\widetilde{M} = [\widetilde{L} \rightarrow G/\text{Ker}^0(g)]$ where \widetilde{L} is the pull-back of L' , the lifting of f is granted by the universal property of pull-backs. In general, we can extend the so obtained isogeny on the image of g to an isogeny of G' (see the proof of Prop. 1.3 in [7] for details). \square

C.4.4. Lemma. *Let*

$$\begin{array}{ccc} M' & \xrightarrow{f} & M \\ u \uparrow & & \uparrow t \\ N' & \xrightarrow{h} & N \end{array}$$

be a commutative diagram in ${}^t\mathcal{M}_1^{\text{eff}}$, where f is strict and u, t are q.i. Then the induced map $v : \text{Coker}(h) \rightarrow \text{Coker}(f)$ is a q.i.

Proof. In all this proof, the term “kernel” is taken in the sense of kernel of complexes of sheaves. Let K and K' be the kernels of f and h respectively:

$$\begin{array}{ccccccc} 0 & \longrightarrow & K & \longrightarrow & M' & \xrightarrow{f} & M \longrightarrow \text{Coker}(f) \\ & & w \uparrow & & u \uparrow & & t \uparrow \\ & & & & & & v \uparrow \\ 0 & \longrightarrow & K' & \longrightarrow & N' & \xrightarrow{h} & N \longrightarrow \text{Coker}(h) \end{array}$$

By a diagram chase, we see that v and $\text{Ker } t \rightarrow \text{Ker } v$ are onto. To conclude, it will be sufficient to show that the sequence of complexes

$$(C.4) \quad \text{Ker}(u) \rightarrow \text{Ker}(t) \rightarrow \text{Ker}(v)$$

is exact termwise. For this, note that the second component of w is onto because f is strict and by dimension reasons. This implies by a diagram chase that the second component of (C.4) is exact. But then the first component has to be exact too. \square

C.5. Exact sequences of 1-motives. We have the following basic properties of 1-motives.

C.5.1. Proposition. *The canonical functor*

$${}^t\mathcal{M}_1^{\text{eff}} \rightarrow {}^t\mathcal{M}_1$$

is left exact and faithful.

Proof. Faithfulness immediately follows from Proposition C.2.3, while left exactness follows from Proposition C.2.4 and (the dual of) [35, Ch. I, Prop. 3.1]. \square

C.5.2. Lemma. *Let $f : M' \rightarrow M$ be an effective map.*

- (1) *The canonical projection $\pi : M \rightarrow \text{Coker}(f)$ remains an epimorphism in ${}^t\mathcal{M}_1[1/p]$.*
- (2) *If f is strict then π remains a cokernel in ${}^t\mathcal{M}_1[1/p]$.*
- (3) *Cokernels exist in ${}^t\mathcal{M}_1[1/p]$.*

Proof. To show Part 1 let $\pi : M \rightarrow N$ be an effective map. One sees immediately that π is epi in ${}^t\mathcal{M}_1[1/p]$ if and only if for any commutative diagram

$$\begin{array}{ccc} M & \xrightarrow{\pi} & N \\ s' \uparrow & & \uparrow s \\ Q' & \xrightarrow{\pi'} & Q \end{array}$$

with s, s' q.i., the map π' is an epi in the effective category. Now specialise to the case $N = \text{Coker}(f)$ and remark that (up to modding out by $\text{Ker } f$) we may assume f to be a monomorphism as a map of complexes, thus strict. Take π', s, s' as above. We have a commutative

diagram of effective maps

$$\begin{array}{ccccccc}
 0 & \longrightarrow & M' & \xrightarrow{f} & M & \xrightarrow{\pi} & \text{Coker}(f) \\
 & & \uparrow s'' & & \uparrow s' & & \uparrow s \\
 & & & & \text{Coker}(f') & & \\
 & & & \nearrow & & \nwarrow u & \\
 Q'' & \xrightarrow{f'} & Q' & \xrightarrow{\pi'} & & & Q.
 \end{array}$$

Here s'' is a q.i. and Q'', f', s'' are obtained by calculus of right fractions (Proposition C.2.4). By Lemma C.4.4, the induced map $t : \text{Coker}(f') \rightarrow \text{Coker}(f)$ is a q.i.. By Proposition C.2.3, $\pi' f' = 0$, hence the existence of u . By Lemma C.2.6, u is a q.i.. Hence π' is a composition of two epimorphisms and Part 1 is proven.

To show Part 2, let $gt^{-1} : M \rightarrow M''$ be such that the composition $M' \rightarrow M''$ is zero. By calculus of right fractions we have a commutative diagram

$$\begin{array}{ccccc}
 M' & \xrightarrow{f} & M & \xrightarrow{\pi} & \text{Coker}(f) \\
 \uparrow u & & \uparrow t & & \uparrow v \\
 N''' & \xrightarrow{h} & N'' & \longrightarrow & \text{Coker}(h) \\
 & & & \searrow g & \downarrow \\
 & & & & M''
 \end{array}$$

where all maps are effective and u is a q.i.. As above we have $gh = 0$, hence the factorisation of g through $\text{Coker}(h)$. Moreover $\text{Coker}(h)$ maps canonically to $\text{Coker}(f)$ via a map v (say), which is a q.i. by Lemma C.4.4. This shows that gt^{-1} factors through $\text{Coker}(f)$ in ${}^t\mathcal{M}_1[1/p]$. Uniqueness of the factorisation is then granted by Part 1.

In a category, the existence of cokernels is invariant by left or right composition by isomorphisms, hence Part 3 is a consequence of Parts 1 and 2 via Proposition C.4.3. \square

Now we can show the following key result (*cf.* [7, Prop. 1.3]).

C.5.3. Theorem. *The category ${}^t\mathcal{M}_1[1/p]$ is abelian.*

Proof. Existence and description of kernels follows from Propositions C.1.3, C.2.4, C.5.1 and (the dual of) [35, Ch. I, Cor. 3.2], while existence of cokernels has been proven in Lemma C.5.2. We are then left to show that, for any (effective) strict map $\varphi : M \rightarrow M'$, the

canonical effective morphism from the coimage of φ to the image of φ is a q.i. of 1-motives, *i.e.* the canonical morphism

$$(C.5) \quad \text{Coker}(\text{Ker } \varphi \rightarrow M) \rightarrow \text{Ker}(M' \rightarrow \text{Coker } \varphi)$$

is a quasi-isomorphism. Since we can split φ in two short exact sequences of complexes in which each term is an effective 1-motive we see that (C.5) is even a isomorphism in ${}^t\mathcal{M}_1^{\text{eff}}[1/p]$. \square

C.5.4. Remark. Note that (even in characteristic zero) for a given non-strict effective map $(f, g) : M \rightarrow M'$ the effective morphism (C.5) is not a q.i. of 1-motives. In fact, the following diagram

$$\begin{array}{ccc} 0 & & 0 \\ \downarrow & & \downarrow \\ \text{Ker}(f)/\text{Ker}^0(f) & \subseteq & \text{Ker}(g)/\text{Ker}^0(g) \\ \downarrow & & \downarrow \\ L/\text{Ker}^0(f) & \longrightarrow & G/\text{Ker}^0(g) \\ \downarrow & & \downarrow \\ \text{Im}(f) & \longrightarrow & \text{Im}(g) \\ \downarrow & & \downarrow \\ 0 & & 0 \end{array}$$

is not a pull-back, in general. For example, let $g : G \rightarrow G'$ be with finite kernel and a proper sub-group $F \subsetneq \text{Ker}(g)$, and consider

$$(0, g) : [F \rightarrow G] \rightarrow [0 \rightarrow G'].$$

C.5.5. Corollary. *A short exact sequence of 1-motives in ${}^t\mathcal{M}_1[1/p]$*

$$(C.6) \quad 0 \rightarrow M' \rightarrow M \rightarrow M'' \rightarrow 0$$

*can be represented up to isomorphisms by a strict effective epimorphism $(f, g) : M \rightarrow M''$ with kernel M' , *i.e.* by an exact sequence of complexes.*

C.5.6. Example. Let M be a 1-motive with torsion. We then always have a canonical exact sequence in ${}^t\mathcal{M}_1[1/p]$

$$(C.7) \quad 0 \rightarrow M_{\text{tor}} \rightarrow M \rightarrow M_{\text{fr}} \rightarrow 0$$

induced by (C.1), according to Definition C.1.2. Note that in the following canonical factorisation

$$\begin{array}{ccc} M & \longrightarrow & M_{\text{fr}} \\ & \searrow \quad \nearrow & \\ & M_{\text{tf}} & \end{array}$$

the effective map $M \rightarrow M_{\text{tf}}$ is a strict epimorphism with kernel M_{tor} and $M_{\text{tf}} \rightarrow M_{\text{fr}}$ is a q.i. (providing an example of Proposition C.4.3).

C.6. ℓ -adic realisation. Let $n : M \rightarrow M$ be the (effective) multiplication by n on a 1-motive $M = [L \xrightarrow{u} G]$ over a field k where n is prime to the characteristic of k . It is then easy to see, *e.g.* by the description of kernels in Proposition C.1.3, that

$${}_nM := \text{Ker}(M \xrightarrow{n} M) = [\text{Ker}(u) \cap {}_nL \rightarrow 0].$$

Thus ${}_nM = 0$ (all n in characteristic zero) if and only if M is torsion-free, *i.e.* $M_{\text{tor}} = 0$. Moreover, by Proposition C.4.3 and Lemma C.5.2, we can see that

$$M/n := \text{Coker}(M \xrightarrow{n} M)$$

is always a torsion 1-motive. If $L = 0$, let simply G denote, as usual, the 1-motive $[0 \rightarrow G]$. Then we get an extension in $\mathcal{M}_1[1/p]$

$$(C.8) \quad 0 \rightarrow G \xrightarrow{n} G \rightarrow {}_nG[1] \rightarrow 0$$

where ${}_nG[1]$ is the torsion 1-motive $[{}_nG \rightarrow 0]$. If $L \neq 0$ then M/n can be regarded as an extension of L/n by $\text{Coker}({}_nL \rightarrow {}_nG)$, *e.g.* also by applying the snake lemma to the multiplication by n on the following canonical short exact sequence (here $L[1] = [L \rightarrow 0]$ as usual)

$$(C.9) \quad 0 \rightarrow G \rightarrow M \rightarrow L[1] \rightarrow 0$$

of effective 1-motives (which is also exact in $\mathcal{M}_1[1/p]$ by Corollary C.5.5). Summarizing up from (C.8), (C.9) we then get a long exact sequence in ${}^t\mathcal{M}_1^{\text{tor}}[1/p]$

$$(C.10) \quad 0 \rightarrow {}_nM \rightarrow {}_nL[1] \rightarrow {}_nG[1] \rightarrow M/n \rightarrow L/n[1] \rightarrow 0.$$

Let now be $n = \ell^\nu$ where $\ell \neq \text{char}(k)$. Set:

C.6.1. Definition. The ℓ -adic realisation of a 1-motive M is

$$T_\ell(M) := \varprojlim_{\nu} L_\nu$$

in the category of ℓ -adic sheaves, where $M/\ell^\nu = [L_\nu \rightarrow 0]$.

Since the inverse system “ \varprojlim_{ν} ” $_{\nu} L$ is Mittag-Leffler trivial, we obtain a short exact sequence

$$0 \rightarrow T_{\ell}(G) \rightarrow T_{\ell}(M) \rightarrow L \otimes \mathbb{Z}_{\ell} \rightarrow 0$$

where $T_{\ell}(G)$ is the Tate module of the semiabelian variety G . More generally, using Corollary C.5.5, we have:

C.6.2. Lemma. *The functor T_{ℓ} is exact on ${}^t\mathcal{M}_1[1/p]$, and extends canonically to ${}^t\mathcal{M}_1 \otimes \mathbb{Z}_{\ell}$. \square*

C.7. Deligne 1-motives. Let ${}^t\mathcal{M}_1^{\text{fr}}[1/p]$, ${}^t\mathcal{M}_1^{\text{tor}}[1/p]$ and ${}^t\mathcal{M}_1^{\text{tf}}[1/p]$ denote the corresponding full sub-categories of ${}^t\mathcal{M}_1[1/p]$ given by free, torsion and torsion-free effective 1-motives respectively. The following $M \mapsto M_{\text{fr}}$ (resp. $M \mapsto M_{\text{tor}}$) define functors from ${}^t\mathcal{M}_1[1/p]$ to ${}^t\mathcal{M}_1^{\text{fr}}[1/p]$ (resp. from ${}^t\mathcal{M}_1[1/p]$ to ${}^t\mathcal{M}_1^{\text{tor}}[1/p]$). We have (cf. [7, (1.1.3)]):

C.7.1. Proposition. *The natural functor*

$$\mathcal{M}_1[1/p] \rightarrow {}^t\mathcal{M}_1[1/p]$$

from Deligne 1-motives to 1-motives with torsion has a left adjoint/left inverse given by $M \mapsto M_{\text{fr}}$. In particular, it is fully faithful and makes $\mathcal{M}_1[1/p]$ an exact sub-category of ${}^t\mathcal{M}_1[1/p]$. The above left adjoint defines equivalences

$$\mathcal{M}_1[1/p] \cong {}^t\mathcal{M}_1^{\text{fr}}[1/p] \cong {}^t\mathcal{M}_1^{\text{tf}}[1/p].$$

Proof. Consider an effective map $(f, g) : \widetilde{M} \rightarrow M'$, to a free 1-motive M' , and a q.i. $\widetilde{M} \rightarrow M$, i.e. $M = [\widetilde{L}/F \rightarrow \widetilde{G}/F]$ for a finite group F . Since M' is free then F is contained in the kernel of f and the same holds for g . Thus (f, g) induces an effective map $M \rightarrow M'$. Let $M = [L \xrightarrow{u} G]$. Then $L_{\text{tor}} \subseteq \text{Ker}(f)$ and also $u(L_{\text{tor}}) \subseteq \text{Ker}(g)$ yielding an effective map $(f, g) : M_{\text{fr}} \rightarrow M'$. This proves the first assertion.

Since ${}^t\mathcal{M}_1^{\text{eff,fr}} \hookrightarrow {}^t\mathcal{M}_1^{\text{eff,tf}}$, the claimed equivalence is obtained from the canonical q.i. $M \rightarrow M_{\text{fr}}$ for $M \in {}^t\mathcal{M}_1^{\text{eff,tf}}$, see (C.1). Finally, consider the exact sequence (C.6) of 1-motives with torsion such that $M'_{\text{tor}} = M''_{\text{tor}} = 0$. Since M_{tor} is mapped to zero in M'' , it injects in M' . Thus also M is torsion-free, i.e. $M = M_{\text{tf}}$, and quasi-isomorphic to M_{fr} . \square

C.7.2. Remark. We also clearly have that the functor $M \mapsto M_{\text{tor}}$ is a right-adjoint to the embedding ${}^t\mathcal{M}_1^{\text{tor}}[1/p] \hookrightarrow {}^t\mathcal{M}_1[1/p]$, i.e.

$$\text{Hom}_{\text{eff}}(M, M'_{\text{tor}}) \cong \text{Hom}(M, M')$$

for $M \in {}^t\mathcal{M}_1^{\text{tor}}[1/p]$ and $M' \in {}^t\mathcal{M}_1[1/p]$.

C.7.3. Corollary. *We have ${}^t\mathcal{M}_1^{\text{tor}} \otimes \mathbb{Q} = 0$ and the full embedding $\mathcal{M}_1[1/p] \rightarrow {}^t\mathcal{M}_1[1/p]$ induces an equivalence*

$$\mathcal{M}_1 \otimes \mathbb{Q} \xrightarrow{\sim} {}^t\mathcal{M}_1 \otimes \mathbb{Q}.$$

C.8. Non-connected 1-motives. We consider a larger category allowing non-connected (reduced) group schemes as a supplement of Proposition 1.1.5.

C.8.1. Definition. Let $\mathcal{M}_{\text{nc}}^{\text{eff}}$ denote the following category. The objects are $N = [L \rightarrow G]$ complexes of étale sheaves over the field k where L is discrete and G is a reduced group scheme locally of finite type over k such that

- (i) the connected component of the identity G^0 is semiabelian, and
- (ii) $\pi_0(G)$ is finitely generated.

The morphisms are just maps of complexes. We call $\mathcal{M}_{\text{nc}}^{\text{eff}}$ the category of *effective non-connected* 1-motives.

We denote $\mathcal{M}_{\text{anc}}^{\text{eff}}$ the full subcategory of $\mathcal{M}_{\text{nc}}^{\text{eff}}$ whose objects are $N = [L \rightarrow G]$ as above such that G is of finite type over k (then condition (ii) is automatically granted and $\pi_0(G)$ is a finite group scheme). We call $\mathcal{M}_{\text{anc}}^{\text{eff}}$ the category of *algebraic effective non-connected* 1-motives.

Note that a representable presheaf on the category of schemes over k can be characterised by axiomatic methods, including Condition (i) above, *cf.* the Appendix in [8].

Associated to $N = [L \rightarrow G]$ we have the following diagram

$$\begin{array}{ccc} 0 & & 0 \\ \downarrow & & \downarrow \\ L^0 & \longrightarrow & G^0 \\ \downarrow & & \downarrow \\ L & \longrightarrow & G \\ \downarrow & & \downarrow \\ L/L^0 & \xrightarrow{\subseteq} & \pi_0(G) \\ \downarrow & & \downarrow \\ 0 & & 0 \end{array}$$

here L^0 denote the pull-back of G^0 along $L \rightarrow G$. Let

$$N^0 := [L^0 \rightarrow G^0]$$

denote the effective 1-motive associated to N and denote

$$\pi_0(N) := [L/L^0 \hookrightarrow \pi_0(G)].$$

We say that $D = [L \hookrightarrow L']$ is *discrete* if L' is a discrete sheaf and L injects into L' .

Denote $\mathcal{M}_{\text{nc}}^{\text{dis}}$ the full subcategory of $\mathcal{M}_{\text{nc}}^{\text{eff}}$ given by discrete objects. Note that N is discrete if and only if $\pi_0(N) = N$ (if and only if $N^0 = 0$).

C.8.2. Proposition. *The operation $N \mapsto N^0$ defines a functor*

$$c^{\text{eff}} : \mathcal{M}_{\text{nc}}^{\text{eff}} \rightarrow {}^t\mathcal{M}_1^{\text{eff}}$$

which is right adjoint to the embedding $i_1 : {}^t\mathcal{M}_1^{\text{eff}} \hookrightarrow \mathcal{M}_{\text{nc}}^{\text{eff}}$ and $c^{\text{eff}}i_1 = 1$. Moreover, we have a functor

$$\pi_0 : \mathcal{M}_{\text{nc}}^{\text{eff}} \rightarrow \mathcal{M}_{\text{nc}}^{\text{dis}}$$

which is left adjoint to $i_{\text{dis}} : \mathcal{M}_{\text{nc}}^{\text{dis}} \hookrightarrow {}^t\mathcal{M}_1^{\text{eff}}$ and $\pi_0i_{\text{dis}} = 1$.

Proof. Straightforward. \square

C.8.3. Remarks. 1) The same results as in Proposition C.8.2 above refine to $\mathcal{M}_{\text{anc}}^{\text{eff}}$ and $\mathcal{M}_{\text{anc}}^{\text{dis}}$.

2) Note that $\mathcal{M}_{\text{nc}}^{\text{eff}}$ has kernels. Let $\varphi = (f, g) : N \rightarrow N'$ be a map in $\mathcal{M}_{\text{nc}}^{\text{eff}}$. Let $g^0 : G^0 \rightarrow G'^0$ and $\pi_0(g) : \pi_0(G) \rightarrow \pi_0(G')$ be the induced maps. Then $\text{Ker}(\varphi) = [\text{Ker}(f) \rightarrow \text{Ker}(g)]$ as a complex of sheaves; in fact $\text{Ker}(g)$ is representable, $\text{Ker}^0(g^0) = (\text{Ker}(g))^0$ and $\pi_0(\text{Ker}(g))$ maps to $\text{Ker}(\pi_0(g))$ with finite kernel. However, it is easy to see that $\mathcal{M}_{\text{nc}}^{\text{eff}}$ is not abelian.

C.8.4. Proposition. *The category $\mathcal{M}_{\text{anc}}^{\text{eff}}[1/p]$ is abelian.*

Proof. Regard $\mathcal{M}_{\text{anc}}^{\text{eff}}[1/p]$ as a full subcategory of $C^{[-1,0]}(\text{Shv}(k_{\text{ét}}))[1/p]$ of the abelian category of complexes of sheaves concentrated in degree -1 and 0 . For a map $\varphi = (f, g) : N \rightarrow N'$, $\text{Ker}(\varphi) = [\text{Ker}(f) \rightarrow \text{Ker}(g)] \in \mathcal{M}_{\text{anc}}^{\text{eff}}$ and $\text{Coker}(\varphi) = [\text{Coker}(f) \rightarrow \text{Coker}(g)] \in \mathcal{M}_{\text{anc}}^{\text{eff}}$. For an extension $0 \rightarrow N \rightarrow N' \rightarrow N'' \rightarrow 0$ in $C^{[-1,0]}(\text{Shv}(k_{\text{ét}}))$ such that N and N'' belongs to $\mathcal{M}_{\text{anc}}^{\text{eff}}$ then also $N' \in \mathcal{M}_{\text{anc}}^{\text{eff}}$. \square

C.9. Homs and Extensions. We will provide a characterisation of the Yoneda Ext in the abelian category ${}^t\mathcal{M}_1[1/p]$.

C.9.1. Proposition. *We have*

- (a) $\text{Hom}_{{}^t\mathcal{M}_1}(L[1], L'[1]) = \text{Hom}_k(L, L')$,
- (b) $\text{Hom}_{{}^t\mathcal{M}_1}(L[1], G') = 0$,
- (c) $\text{Hom}_{{}^t\mathcal{M}_1}(G, G') \subseteq \text{Hom}_k(A, A') \times \text{Hom}_k(T, T')$ if G (resp. G') is an extension of an abelian variety A by a torus T (resp. of A' by T'),

(d) $\mathrm{Hom}_{t\mathcal{M}_1}(G, L'[1]) = \mathrm{Hom}_k({}_nG, L'_{\mathrm{tor}})$ if $nL'_{\mathrm{tor}} = 0$.

In particular, the group $\mathrm{Hom}_{t\mathcal{M}_1}(M, M')$ is finitely generated for all 1-motives $M, M' \in {}^t\mathcal{M}_1[1/p]$.

Proof. Since there are no q.i. to $L[1]$, we have $\mathrm{Hom}_{t\mathcal{M}_1}(L[1], L'[1]) = \mathrm{Hom}_{\mathrm{eff}}(L[1], L'[1])$ and the latter is clearly isomorphic to $\mathrm{Hom}_k(L, L')$. By Proposition C.7.1, we have $\mathrm{Hom}_{t\mathcal{M}_1}(L[1], G') = \mathrm{Hom}_{\mathrm{eff}}(L[1], G')$ and $\mathrm{Hom}_{t\mathcal{M}_1}(G, G') = \mathrm{Hom}_{\mathrm{eff}}(G, G')$. The former is clearly 0 while $\mathrm{Hom}_{\mathrm{eff}}(G, G') = \mathrm{Hom}_k(G, G') \subseteq \mathrm{Hom}_k(A, A') \times \mathrm{Hom}_k(T, T')$ since $\mathrm{Hom}_k(T, A') = \mathrm{Hom}_k(A, T') = 0$. For (d), let $[F \rightarrow \tilde{G}] \rightarrow [0 \rightarrow G]$ be a q.i. and $[F \rightarrow \tilde{G}] \rightarrow [L' \rightarrow 0]$ be an effective map providing an element of $\mathrm{Hom}(G, L'[1])$. If L' is free then it yields the zero map, as F is torsion. Thus $\mathrm{Hom}_{t\mathcal{M}_1}(G, L'[1]) = \mathrm{Hom}_{t\mathcal{M}_1}(G, L'_{\mathrm{tor}}[1])$. For $n \in \mathbb{N}$ consider the short exact sequence (C.8) in ${}^t\mathcal{M}_1$. If n is such that $nL'_{\mathrm{tor}} = 0$ taking $\mathrm{Hom}_{t\mathcal{M}_1}(-, L'_{\mathrm{tor}}[1])$ we further obtain $\mathrm{Hom}_{t\mathcal{M}_1}(G, L'[1]) = \mathrm{Hom}_k({}_nG, L'_{\mathrm{tor}})$.

The last statement follows from these computations and an easy dévissage from (C.9). \square

C.9.2. Remark. If we want to get rid of the integer n in (d), we may equally write

$$\mathrm{Hom}_{t\mathcal{M}_1}(G, L'[1]) = \mathrm{Hom}_{\mathrm{cont}}(\widehat{T}(G), L'_{\mathrm{tor}}) = \mathrm{Hom}_{\mathrm{cont}}(\widehat{T}(G), L')$$

where $\widehat{T}(G) = \prod_{\ell} T_{\ell}(G)$ is the complete Tate module of G .

C.9.3. Proposition. *We have isomorphisms (for Ext in ${}^t\mathcal{M}_1[1/p]$):*

- (a) $\mathrm{Ext}_k^1(L, L') \xrightarrow{\sim} \mathrm{Ext}_{t\mathcal{M}_1}^1(L[1], L'[1]),$
- (b) $\mathrm{Hom}_k(L, G') \xrightarrow{\sim} \mathrm{Ext}_{t\mathcal{M}_1}^1(L[1], G'),$
- (c) $\mathrm{Ext}_k^1(G, G') \xrightarrow{\sim} \mathrm{Ext}_{t\mathcal{M}_1}^1(G, G')$ and
- (d) $\varinjlim_n \mathrm{Ext}_k^1({}_nG, L') \xrightarrow{\sim} \mathrm{Ext}_{t\mathcal{M}_1}^1(G, L'[1]);$ these two groups are 0 if L' is torsion and k algebraically closed.

Proof. By Corollary C.5.5, any short exact sequence of 1-motives can be represented up to isomorphism by a short exact sequence of complexes in which each term is an effective 1-motive.

For (a), just observe that there are no nontrivial q.i. of 1-motives with zero semiabelian part. For (b), note that an extension of $L[1]$ by G' is given by a diagram

$$(C.11) \quad \begin{array}{ccccccc} 0 & \longrightarrow & F' & \longrightarrow & L'' & \longrightarrow & L \longrightarrow 0 \\ & & \downarrow & & \downarrow v & & \downarrow \\ 0 & \longrightarrow & \tilde{G}' & \longrightarrow & G'' & \longrightarrow & 0 \longrightarrow 0 \end{array}$$

where $\widetilde{M}' = [F' \rightarrow \widetilde{G}']$ is q.i. to $[0 \rightarrow G']$. When $F' = 0$, this diagram is equivalent to the datum of v : this provides a linear map $\mathrm{Hom}_k(L, G') \rightarrow \mathrm{Ext}_{t\mathcal{M}_1}^1(L[1], G')$. This map is surjective since we may always mod out by F' in (C.11) and get an quasi-isomorphic exact sequence with $F' = 0$. It is also injective: if (C.11) (with $F' = 0$) splits in ${}^t\mathcal{M}_1[1/p]$, it already splits in ${}^t\mathcal{M}_1^{\mathrm{eff}}[1/p]$ and then $v = 0$.

For (c) we see that an extension of G by G' in ${}^t\mathcal{M}_1[1/p]$ can be represented by a diagram

$$\begin{array}{ccccccccc} 0 & \longrightarrow & F' & \longrightarrow & L'' & \longrightarrow & F & \longrightarrow & 0 \\ & & \downarrow & & \downarrow & & \downarrow & & \\ 0 & \longrightarrow & \widetilde{G}' & \longrightarrow & G'' & \longrightarrow & \widetilde{G} & \longrightarrow & 0 \end{array}$$

with \widetilde{M}' as in (b) and $\widetilde{M} = [F \rightarrow \widetilde{G}]$ q.i. to $[0 \rightarrow G]$. Since the top line is exact, L'' is finite. For $F = F' = 0$ we just get a group scheme extension of G by G' , hence a homomorphism $\mathrm{Ext}_k^1(G, G') \rightarrow \mathrm{Ext}_{t\mathcal{M}_1}^1(G, G')$. This homomorphism is surjective: dividing by F' we get a quasi-isomorphic exact sequence

$$\begin{array}{ccccccccc} 0 & \longrightarrow & 0 & \longrightarrow & L''/F' & \xrightarrow{\sim} & F & \longrightarrow & 0 \\ & & \downarrow & & \downarrow & & \downarrow & & \\ 0 & \longrightarrow & G' & \longrightarrow & G''/F' & \longrightarrow & \widetilde{G} & \longrightarrow & 0 \end{array}$$

and further dividing by F we then obtain

$$\begin{array}{ccccccccc} 0 & \longrightarrow & 0 & \longrightarrow & 0 & \longrightarrow & 0 & \longrightarrow & 0 \\ & & \downarrow & & \downarrow & & \downarrow & & \\ 0 & \longrightarrow & G' & \longrightarrow & G''/L'' & \longrightarrow & G & \longrightarrow & 0. \end{array}$$

Injectivity is seen as in (b).

For (d) we first construct a map $\Phi_n : \mathrm{Ext}_k^1({}_nG, L') \rightarrow \mathrm{Ext}_{t\mathcal{M}_1}^1(G, L'[1])$ for all n . Let $[L''] \in \mathrm{Ext}_k({}_nG, L')$ and consider the following diagram

$$(C.12) \quad \begin{array}{ccccccccc} 0 & \longrightarrow & L' & \longrightarrow & L'' & \longrightarrow & {}_nG & \longrightarrow & 0 \\ & & \downarrow & & \downarrow & & \downarrow & & \\ 0 & \longrightarrow & 0 & \longrightarrow & G & \xlongequal{\quad} & G & \longrightarrow & 0. \end{array}$$

Since $[{}_nG \rightarrow G]$ is q.i. to $[0 \rightarrow G]$, this provides an extension of G by $L'[1]$ in ${}^t\mathcal{M}_1[1/p]$. For n variable $\{\mathrm{Ext}_k^1({}_nG, L')\}_n$ is a direct system and one checks easily that the maps Φ_n are compatible (by pull-back), yielding a well-defined linear map

$$\Phi : \varinjlim \mathrm{Ext}_k^1({}_nG, L') \rightarrow \mathrm{Ext}_{t\mathcal{M}_1}^1(G, L'[1]).$$

This map is surjective since any extension of G by $L'[1]$ can be represented by a diagram (C.12) for some n (as multiplication by n is cofinal in the direct system of isogenies). We now show that Φ is also injective.

Let $n \mid m$, e.g. $rn = m$, so that the following sequence is exact

$$0 \rightarrow {}_rG \rightarrow {}_mG \xrightarrow{r} {}_nG \rightarrow 0$$

and yields a long exact sequence

$$\mathrm{Hom}_k({}_mG, L') \rightarrow \mathrm{Hom}_k({}_rG, L') \rightarrow \mathrm{Ext}_k^1({}_nG, L') \xrightarrow{r} \mathrm{Ext}_k^1({}_mG, L').$$

If L' is torsion, we have $rL' = 0$ for some r , hence $\varinjlim \mathrm{Ext}_k^1({}_nG, L') = 0$ if k is algebraically closed. This shows in particular that $\mathrm{Ext}_{t\mathcal{M}_1}^1(G, L'[1]) = 0$ in this case.

Suppose now that L' is free. Then we have $\mathrm{Hom}_k({}_rG, L') = 0$, hence the transition maps are injective. Therefore, to check that Φ is injective it suffices to check that Φ_n is injective for all n .

Let $\sigma : G \rightarrow [L'' \rightarrow G]$ be a section of (C.12) in ${}^t\mathcal{M}_1[1/p]$. Then σ can be represented by a diagram of effective maps

$$\begin{array}{ccccccc} & & & & {}_{rn}G & & \\ & & & \swarrow & \downarrow & \searrow & \\ & & & & G & & \\ 0 & \longrightarrow & L' & \longrightarrow & L'' & \longrightarrow & {}_nG \\ & & \downarrow & & \downarrow & & \downarrow \\ & & & & G & & \\ & & & \swarrow & & \searrow & \\ 0 & \longrightarrow & 0 & \longrightarrow & G & \xlongequal{\quad} & G \end{array}$$

for some r , where the southwest map is a q.i. To say that σ is a section is to say that this diagram commutes. Hence the image of ${}_{rn}G$ in L'' surjects onto ${}_nG$, and it also injects since L' is torsion-free. This means that the projection $L'' \rightarrow {}_nG$ has a section, hence $[L''] = 0$ in $\mathrm{Ext}_k^1({}_nG, L'')$.

For a general L' , we reduce to these two special cases through an easy diagram chase. \square

C.10. Projective objects in ${}^t\mathcal{M}_1[1/p]$. We show that there are not enough projective objects in ${}^t\mathcal{M}_1[1/p]$, at least when k is algebraically closed:

C.10.1. Proposition. *Suppose that $k = \bar{k}$. Then the only projective object of ${}^t\mathcal{M}_1[1/p]$ is 0.*

Proof. Suppose that $M = [L \rightarrow G] \in {}^t\mathcal{M}_1[1/p]$ is such that $\text{Ext}(M, N) = 0$ for any $N \in {}^t\mathcal{M}_1[1/p]$. From (C.9) we then get a long exact sequence

$$\begin{aligned} \text{Hom}(G, \mathbb{G}_m) &\rightarrow \text{Ext}(L[1], \mathbb{G}_m) \rightarrow \text{Ext}(M, \mathbb{G}_m) \\ &\rightarrow \text{Ext}(G, \mathbb{G}_m) \rightarrow \text{Ext}^2(L[1], \mathbb{G}_m) \end{aligned}$$

where $\text{Ext}(M, \mathbb{G}_m) = 0$, thus *i)* $\text{Ext}(G, \mathbb{G}_m)$ is finite, and *ii)* $\text{Hom}(L, \mathbb{G}_m)$ is finitely generated. We also have an exact sequence

$$\text{Hom}(T, \mathbb{G}_m) \rightarrow \text{Ext}(A, \mathbb{G}_m) \rightarrow \text{Ext}(G, \mathbb{G}_m)$$

where $\text{Hom}(T, \mathbb{G}_m)$ is the character group of the torus T and $\text{Ext}(A, \mathbb{G}_m)$ is the group of k -points of the dual abelian variety A , the abelian quotient of G . From *i)* we get $A = 0$. Since $\text{Hom}(L, \mathbb{G}_m)$ is an extension of a finite group by a divisible group, from *ii)* we get that L is a finite group. Now consider the exact sequence, for $l \neq p$

$$\begin{aligned} 0 \rightarrow \text{Hom}(L[1], \mathbb{Z}/l[1]) &\rightarrow \text{Hom}(M, \mathbb{Z}/l[1]) \\ &\rightarrow \text{Hom}(T, \mathbb{Z}/l[1]) \rightarrow \text{Ext}(L[1], \mathbb{Z}/l[1]) \rightarrow 0 \end{aligned}$$

where the right-end vanishing is $\text{Ext}(M, \mathbb{Z}/l[1]) = 0$ by assumption. Now $\text{Hom}(T, \mathbb{Z}/l[1]) = \text{Ext}(T, \mathbb{Z}/l)$ and any extension of the torus T is lifted to an extension of M by \mathbb{Z}/l , therefore to an element of $\text{Hom}(M, \mathbb{Z}/l[1]) = \text{Ext}(M, \mathbb{Z}/l)$. This yields $\text{Ext}(L, \mathbb{Z}/l) = 0$ for any prime $l \neq p$, thus we see that $L = 0$.

Finally, $[0 \rightarrow \mathbb{G}_m]$ is not projective since, for $n > 1$, the epimorphism

$$[0 \rightarrow \mathbb{G}_m] \xrightarrow{n} [0 \rightarrow \mathbb{G}_m]$$

is not split. \square

C.11. Weights. If $M = [L \xrightarrow{u} G] \in {}^t\mathcal{M}_1[1/p]$ is free then Deligne [25] equipped $M = M_{\text{fr}}$ with an increasing filtration by sub-1-motives as follows:

$$W_{-2}(M) := [0 \rightarrow T] \subseteq W_{-1}(M) := [0 \rightarrow G] \subseteq W_0(M) := M$$

If M is torsion-free we then pull-back the weight filtration along the effective map $M \rightarrow M_{\text{fr}}$ as follows:

$$W_i(M) := \begin{cases} M & i \geq 0 \\ [L_{\text{tor}} \hookrightarrow G] & i = -1 \\ [L_{\text{tor}} \cap T \hookrightarrow T] & i = -2 \\ 0 & i \leq -3 \end{cases}$$

Note that $W_i(M)$ is q.i. to $W_i(M_{\text{fr}})$.

If M has torsion we then further pull-back the weight filtration along the effective map $M \rightarrow M_{\text{tf}}$.

C.11.1. Definition. Let $M = [L \xrightarrow{u} G]$ be an effective 1-motive. Let $u_A : L \rightarrow A$ denote the induced map where $A = G/T$. Define

$$W_i(M) := \begin{cases} M & i \geq 0 \\ [L_{\text{tor}} \rightarrow G] & i = -1 \\ [L_{\text{tor}} \cap \text{Ker}(u_A) \rightarrow T] & i = -2 \\ M_{\text{tor}} = L_{\text{tor}} \cap \text{Ker}(u)[1] & i = -3 \\ 0 & i \leq -4 \end{cases}$$

C.11.2. Remark. It is easy to see that $M \mapsto W_i(M)$ yields a functor from ${}^t\mathcal{M}_1[1/p]$ to ${}^t\mathcal{M}_1[1/p]$. However, this does not define a weight filtration on ${}^t\mathcal{M}_1[1/p]$ in the sense of Definition E.7.2 or Remark E.7.8.

C.12. 1-motives over a base. Let S be a scheme. According to [25, (10.1.10)], a smooth 1-motive over S is a complex $[L \rightarrow G]$ of S -group schemes where L is a lattice (corresponding to a locally constant \mathbb{Z} -constructible free étale sheaf) and G is an extension of an abelian scheme A by a torus T . By [34, Cor. 2.11], it is sufficient to have this condition fibre by fibre as long as the rank of T_s is locally constant. Smooth 1-motives form an additive category denoted (here) by $\mathcal{M}_1(S)$. A smooth 1-motive is provided with a weight filtration as over a field.

By [34, loc. cit.] and [21], the same arguments as in the proofs of Propositions 1.1.5 and 15.2.1 yield

C.12.1. Proposition. *Suppose S regular connected and essentially of finite type over a field k . Then the category $\mathcal{M}_1(S) \otimes \mathbb{Q}$ is abelian and is provided with a 3 step weight filtration in the sense of Definition E.7.2, such that the full subcategories of objects of pure weight are semi-simple.*

C.12.2. Lemma. *In Proposition C.12.1, we have a canonical monomorphism*

$$\text{Ext}^2([0 \rightarrow \mathbb{G}_m], [\mathbb{Z} \rightarrow 0]) \hookrightarrow \text{Ext}_S^1(\mathbb{Z}, \mathbb{G}_m) \otimes \mathbb{Q}.$$

Proof. Let

$$\begin{array}{ccccccc} 0 & \longrightarrow & L_1 & \longrightarrow & L_2 & \longrightarrow & \mathbb{Z} \\ \downarrow & & \downarrow & & \downarrow & & \downarrow \\ \mathbb{G}_m & \longrightarrow & G_1 & \longrightarrow & G_2 & \longrightarrow & 0 \end{array}$$

be a 2-extension E of $[\mathbb{Z} \rightarrow 0]$ by $[0 \rightarrow \mathbb{G}_m]$. Taking Tot, we get a short exact sequence of sheaves

$$0 \rightarrow L_1 \oplus \mathbb{G}_m \rightarrow L_2 \oplus G_1 \rightarrow \mathbb{Z} \oplus G_2 \rightarrow 0.$$

We define the image of E in $\mathrm{Ext}_S^1(\mathbb{Z}, \mathbb{G}_m)$ as the image of this extension E' under the composition

$$\mathrm{Ext}_S^1(\mathbb{Z} \oplus G_2, L_1 \oplus \mathbb{G}_m) \twoheadrightarrow \mathrm{Ext}_S^1(\mathbb{Z} \oplus G_2, \mathbb{G}_m) \twoheadrightarrow \mathrm{Ext}_S^1(\mathbb{Z}, \mathbb{G}_m).$$

It remains to see the injectivity. If the image of E is 0, a diagram chase shows that E' is in the image of

$$\mathrm{Ext}_S^1(\mathbb{Z} \oplus G_2, L_1) \oplus \mathrm{Ext}_S^1(G_2, L_1 \oplus \mathbb{G}_m) \rightarrow \mathrm{Ext}_S^1(\mathbb{Z} \oplus G_2, L_1 \oplus \mathbb{G}_m).$$

Since S is regular, both groups are 0 rationally: for the vanishing of $\mathrm{Ext}_S^1(G_2, L_1)$, observe that Proposition 3.1.5 works over S , with the same proof. Thus E' is rationally split. Since $\mathrm{Hom}(G_2, L_1) = 0$, this amounts to \mathbb{Q} -splittings of $G_1 \rightarrow G_2$ and $L_2 \rightarrow \mathbb{Z}$ plus a \mathbb{Q} -morphism $\mathbb{Z} \rightarrow G_1$ with a compatibility condition. After trivialising the splittings, we get a diagram

$$\begin{array}{ccccccc} 0 & \longrightarrow & L_1 & \longrightarrow & L_1 \oplus \mathbb{Z} & \longrightarrow & \mathbb{Z} \\ \downarrow & & f \downarrow & & g \downarrow & & \downarrow \\ \mathbb{G}_m & \longrightarrow & \mathbb{G}_m \oplus G_2 & \longrightarrow & G_2 & \longrightarrow & 0 \end{array}$$

and the compatibility condition amounts to $g|_{L_1} = 0$ rationally. Then f factors rationally through \mathbb{G}_m . Thus E is (rationally) the direct sum of a 2-extension of 0 by $[0 \rightarrow \mathbb{G}_m]$ and a 2-extension of $[\mathbb{Z} \rightarrow 0]$ by 0, hence is 0. \square

C.12.3. Proposition. *In Proposition C.12.1, assume S semi-local. Then $\mathcal{M}_1(S) \otimes \mathbb{Q}$ is of cohomological dimension ≤ 1 .*

Proof. By Proposition C.12.1 and Lemma E.7.5, we are left to show that $\mathrm{Ext}^2(M, N) = 0$ for M pure of weight 0 and N pure of weight -2 . Since we work with rational coefficients, we may by a transfer argument reduce to $M = [\mathbb{Z} \rightarrow 0]$, $N = [0 \rightarrow \mathbb{G}_m]$. Then

$$\mathrm{Ext}^2(M, N) \hookrightarrow \mathrm{Ext}_S^1(\mathbb{Z}, \mathbb{G}_m) \otimes \mathbb{Q} = H_{\mathrm{\acute{e}t}}^1(S, \mathbb{G}_m) \otimes \mathbb{Q}$$

by Lemma C.12.2, and the latter group is 0 because S is semi-local and regular. \square

APPENDIX D. HOMOTOPY INVARIANCE FOR ÉTALE SHEAVES WITH TRANSFERS

One of the main results of Voevodsky concerning presheaves with transfers is that, over a perfect field k , a Nisnevich sheaf with transfers F is homotopy invariant (that is, $F(X) \xrightarrow{\sim} F(X \times \mathbb{A}^1)$ for any smooth X) if and only if it is strongly homotopy invariant, that is, $H_{\mathrm{Nis}}^i(X, F) \xrightarrow{\sim} H_{\mathrm{Nis}}^i(X \times \mathbb{A}^1, F)$ for any smooth X and any $i \geq 0$. This allows him to define the *homotopy t -structure* on $\mathrm{DM}_-^{\mathrm{eff}}$.

These results remain “as true as can be” in the étale topology, at least if k has finite étale cohomological dimension. According to an established tradition, this result is probably well-known to experts but we haven’t been able to find it in the literature: it could have been formulated and proven for example in [61]. The aim of this appendix is to provide proofs, for which our main source of results will be [61].

D.1. Homotopy invariance and strict homotopy invariance.

D.1.1. Definition. We denote as in [61, Def. 2.1] by $\text{PST}(k) = \text{PST}$ the category of presheaves with transfers on smooth k -varieties. We also denote by $\text{EST}(k)$, or simply EST , the category of étale sheaves with transfers over k .

According to [61, Def. 2.15 and 9.22]:

D.1.2. Definition. a) An object F of PST or EST is *homotopy invariant* if $F(X) \xrightarrow{\sim} F(X \times \mathbb{A}^1)$ for any smooth k -variety X .

b) Let $F \in \text{EST}$. Then F is *strictly homotopy invariant* if $H_{\text{ét}}^i(X, F) \xrightarrow{\sim} H_{\text{ét}}^i(X \times \mathbb{A}^1, F)$ for any smooth k -variety X and any $i \geq 0$.

We denote by $\text{HI}_{\text{ét}}(k) = \text{HI}_{\text{ét}}$ the full subcategory of EST consisting of homotopy invariant sheaves, and by $\text{HI}_{\text{ét}}^s(k) = \text{HI}_{\text{ét}}^s$ the full subcategory of $\text{HI}_{\text{ét}}$ consisting of strictly homotopy invariant sheaves.

(Strict homotopy invariance for F simply means that F is \mathbb{A}^1 -local in $D^-(\text{EST})$, see [61, Lemma 9.24].)

Note that $\text{HI}_{\text{ét}}$ is a thick abelian subcategory of EST : if $0 \rightarrow F' \rightarrow F \rightarrow F'' \rightarrow 0$ is an exact sequence in EST , then $F \in \text{HI}_{\text{ét}}$ if and only if $F', F'' \in \text{HI}_{\text{ét}}$. We shall see below that the same is true for $\text{HI}_{\text{ét}}^s$.

The main example of a sheaf F which is in $\text{HI}_{\text{ét}}$ but not in $\text{HI}_{\text{ét}}^s$ is $F = \mathbb{Z}/p$ in characteristic p : because of the Artin-Schreier exact sequence we have

$$k[t]/\mathcal{P}(k[t]) \xrightarrow{\sim} H_{\text{ét}}^1(\mathbb{A}_k^1, \mathbb{Z}/p)$$

where $\mathcal{P}(x) = x^p - x$.

We are going to show that this captures entirely the obstruction for a sheaf in $\text{HI}_{\text{ét}}$ not to be in $\text{HI}_{\text{ét}}^s$.

The following is an étale analogue of [61, Th. 13.8]:

D.1.3. Lemma. *Let F be a homotopy invariant presheaf with transfers. Suppose moreover that F is a presheaf of $\mathbb{Z}[1/p]$ -modules, where p is the exponential characteristic of k . Then the associated étale sheaf with transfers [61, Th. 6.17] $F_{\text{ét}}$ is strictly homotopy invariant.*

Proof. The following method is classical: let $0 \rightarrow F' \rightarrow F \rightarrow F'' \rightarrow 0$ be an exact sequence of homotopy invariant presheaves with transfers, and consider the corresponding exact sequence $0 \rightarrow F'_{\text{ét}} \rightarrow F_{\text{ét}} \rightarrow F''_{\text{ét}} \rightarrow 0$. If, among $F'_{\text{ét}}$, $F_{\text{ét}}$ and $F''_{\text{ét}}$, two are in $\text{HI}_{\text{ét}}^s$, then clearly so is the third. Using the exact sequence

$$0 \rightarrow F_{\text{tors}} \rightarrow F \rightarrow F \otimes \mathbb{Q} \rightarrow F \otimes \mathbb{Q}/\mathbb{Z} \rightarrow 0$$

for the sheaf F of Lemma D.1.3, this reduces us to the following cases:

- F is a presheaf of \mathbb{Q} -vector spaces. Then the result is true by [61, Lemma 14.25] (reduction to [61, Th. 13.8] by the comparison theorem [61, Prop. 14.23]).
- F is a presheaf of torsion abelian groups. Since, by assumption, this torsion is prime to p , $F_{\text{ét}}$ is locally constant by Suslin-Voevodsky rigidity [61, Th. 7.20]. Then the result follows from [SGA4, XV 2.2] (compare [61, Lemma 9.23]).

□

D.1.4. Proposition. *The inclusion $\text{HI}_{\text{ét}}^s \rightarrow \text{HI}_{\text{ét}}$ has an exact left adjoint given by $F \mapsto F \otimes_{\mathbb{Z}} \mathbb{Z}[1/p]$. In particular, $\text{HI}_{\text{ét}}^s = \text{HI}_{\text{ét}}[1/p]$.*

Proof. The fact that $\text{HI}_{\text{ét}}^s$ is $\mathbb{Z}[1/p]$ -linear follows from the Artin-Schreier exact sequence plus the contractibility of \mathbb{G}_a (compare [86, Prop. 3.3.3 2])). Conversely, Lemma D.1.3 implies that any $\mathbb{Z}[1/p]$ -linear sheaf of $\text{HI}_{\text{ét}}$ belongs to $\text{HI}_{\text{ét}}^s$. The rest of the proposition follows. □

As a complement, let us mention the following proposition, which extends Proposition D.1.4:

D.1.5. Proposition. *Let $F \in \text{HI}_{\text{ét}}$. Then the complex $C_*(F)$ is canonically isomorphic to $F[1/p]$ in $\text{DM}_{-, \text{ét}}^{\text{eff}}$.*

Proof. The map $F \rightarrow F[1/p]$ induces a map $C_*(F) \rightarrow C_*(F[1/p])$. The latter complex is tautologically equal to $C_*(F)[1/p]$. Since $\text{DM}_{-, \text{ét}}^{\text{eff}}$ is $\mathbb{Z}[1/p]$ -linear, the map $C_*(F) \rightarrow C_*(F)[1/p]$ is a quasi-isomorphism. Finally, since $F[1/p] \in \text{HI}_{\text{ét}}^s$ (Prop. D.1.4), the augmentation $F[1/p] \rightarrow C_*(F[1/p])$ is a quasi-isomorphism by [61, Lemma 9.15]. □

D.1.6. Corollary. *Let F be a homotopy invariant Nisnevich sheaf with transfers. Then, the natural functor $\alpha^* : \text{DM}_{-}^{\text{eff}} \rightarrow \text{DM}_{-, \text{ét}}^{\text{eff}}$ sends F to $F_{\text{ét}}[1/p]$.*

Proof. According to [61, Remark 14.3], α^* may be described as the composition

$$\text{DM}_{-}^{\text{eff}} \hookrightarrow D^-(\text{Shv}_{\text{Nis}}(\text{Sm}(k))) \xrightarrow{\alpha^*} D^-(\text{Shv}_{\text{ét}}(\text{Sm}(k))) \xrightarrow{RC} \text{DM}_{-, \text{ét}}^{\text{eff}}$$

where the middle functor is induced by the inverse image functor (change of topology) on sheaves and RC is induced by $K \mapsto C_*(K)$. The result then follows from Proposition D.1.5. \square

D.2. Friendly complexes.

D.2.1. Definition. A object $C \in D^-(\text{EST})$ is *friendly* if there exists an integer $N = N(C)$ such that, for any prime number $l \neq p$, $H_q(C/l) = 0$ for $q > N$ (in other terms, C/l is uniformly bounded below). We denote by $D_{\text{fr}}^-(\text{EST})$ the full subcategory of friendly objects and by $\text{DM}_{\text{fr}, \text{ét}}^{\text{eff}}$ the intersection $D_{\text{fr}}^-(\text{EST}) \cap \text{DM}_{-, \text{ét}}^{\text{eff}}$.

D.2.2. Theorem. $\text{DM}_{\text{gm}, \text{ét}}^{\text{eff}} \subset \text{DM}_{\text{fr}, \text{ét}}^{\text{eff}}$.

Proof. It is clear that $D_{\text{fr}}^-(\text{EST})$ is a thick triangulated subcategory of $D^-(\text{EST})$; hence it suffices to prove that $C_*(L_{\text{ét}}(X))$ is friendly for any smooth scheme X . By [61, Lemmas 6.23 and 9.15], we have for any smooth U , any prime $l \neq p$ and any $q \in \mathbb{Z}$

$$\text{Hom}(C_*(L_{\text{ét}}(X \times U)), \mathbb{Z}/l[q]) \simeq H_{\text{ét}}^q(X \times U, \mathbb{Z}/l)$$

hence

$$\mathcal{E}xt^q(C_*(L_{\text{ét}}(X)), \mathbb{Z}/l) \simeq R_{\text{ét}}^q \pi_* \mathbb{Z}/l$$

where $\pi : X \rightarrow \text{Spec } k$ is the structural morphism. By the cohomological dimension results of [SGA4, Exp. X] and the finiteness results of [SGA4 1/2, Th. finitude], this shows that $\underline{\text{Hom}}_{\text{ét}}(C_*(L_{\text{ét}}(X)), \mathbb{Z}/l)$ is a bounded complex of constructible \mathbb{Z}/l -sheaves. It follows that the biduality morphism

$$C_*(L_{\text{ét}}(X))/l \rightarrow \underline{\text{Hom}}_{\text{ét}}(\underline{\text{Hom}}_{\text{ét}}(C_*(L_{\text{ét}}(X)), \mathbb{Z}/l), \mathbb{Z}/l)$$

is an isomorphism of bounded complexes of constructible \mathbb{Z}/l -sheaves. Moreover, the lower bound is at most $2 \dim X$, hence is independent of l . \square

D.3. The étale homotopy t -structure. The following is an étale analogue of [61, Prop. 14.8]:

D.3.1. Proposition. *Let $K \in D^-(\text{EST})$ be a bounded above complex of étale sheaves with transfers. Suppose either that the étale cohomological dimension of k is finite, or that K is friendly. Then K is \mathbb{A}^1 -local if and only if all its cohomology sheaves are strictly homotopy invariant.*

Proof. In the finite cohomological dimension case, “if” is trivial (cf. [61, Prop. 9.30]). For “only if”, same proof as that of [61, Prop. 14.8], by replacing the reference to [61, Th. 13.8] in *loc. cit.* by a reference

to Lemma D.1.3 (note that if K is \mathbb{A}^1 -local, then it is $\mathbb{Z}[1/p]$ -linear by [86, Prop. 3.3.3 2]) and thus so are its cohomology sheaves).

In the friendly case, note that the two conditions

- \mathbb{A}^1 -local
- having strictly homotopy invariant cohomology sheaves

are stable under triangles: for the first it is obvious and for the second it is because $\mathrm{HI}_{\text{ét}}^s$ is thick in EST by Proposition D.1.4. Considering the exact triangle

$$K \rightarrow K \otimes \mathbb{Q} \rightarrow K \otimes (\mathbb{Q}/\mathbb{Z})' \xrightarrow{+1}$$

we are reduced to show the statement separately for $K \otimes \mathbb{Q}$ and for $K \otimes (\mathbb{Q}/\mathbb{Z})'$. In the first case this works by reduction to Nisnevich cohomology, while in the second case the spectral sequence of [61, Prop. 9.30] also converges, this time because $K \otimes (\mathbb{Q}/\mathbb{Z})'$ is bounded below. \square

D.3.2. Remark. The finite cohomological dimension hypothesis appears in the spectral sequence arguments of the proofs of [61, Prop. 9.30 and 14.8]. We don't know if it is really necessary.¹⁶ Nevertheless, Joël Riou pointed out that this argument trivially extends to fields of virtually finite cohomological dimension: the only issue is for the “if” part, but if we know that an object K is \mathbb{A}^1 -local étale-locally, then it is clearly \mathbb{A}^1 -local. (For example, this covers all fields of arithmetic origin.) Therefore:

D.3.3. Corollary. *If the virtual étale cohomological dimension of k is finite, then $\mathrm{DM}_{-, \text{ét}}^{\text{eff}}$ has a homotopy t -structure, with heart $\mathrm{HI}_{\text{ét}}^s$, and the functor $\alpha^* : \mathrm{DM}_{-, \text{ét}}^{\text{eff}} \rightarrow \mathrm{DM}_{-, \text{ét}}^{\text{eff}}$ is t -exact. Without any cohomological dimension assumption, $\mathrm{DM}_{\text{fr}, \text{ét}}$ has a homotopy t -structure.* \square

APPENDIX E. WEIGHT FILTRATIONS

In this appendix, we propose a theory of weight filtrations adapted to our needs. This is closely related to U. Jannsen's and A. Huber's setting in [46] and [40, 1.2], and fits perfectly with S. Morel's viewpoint in [64]. We relate it precisely to Jannsen's approach in Remark E.7.8. The interested reader is encouraged to check that this theory behaves well with respect to the weight structure on $\mathrm{DM}_{\text{gm}}^{\text{eff}}$ constructed by Bondarko in [19], via the realisation functors considered in Sect. 15.

¹⁶It is not: see [50, C.5], which was devised after the present approach.

I. FILTRATIONS OF ABELIAN CATEGORIES

E.1. A glueing lemma. Let \mathcal{A} be an abelian category. Consider two exact sequences

$$\begin{array}{ccccccc} 0 & \longrightarrow & A' & \longrightarrow & A & \longrightarrow & A'' \longrightarrow 0 \\ 0 & \longrightarrow & B' & \longrightarrow & B & \longrightarrow & B'' \longrightarrow 0 \end{array}$$

and two morphisms $f' : A' \rightarrow B'$, $f'' : A'' \rightarrow B''$. We say that $f : A \rightarrow B$ is a *glueing* of f' and f'' if it yields a commutative diagram of exact sequences.

E.1.1. Lemma. *Let $[A] \in \text{Ext}_{\mathcal{A}}^1(A'', A')$ and $[B] \in \text{Ext}_{\mathcal{A}}^1(B'', B')$ be the extension classes of A and B .*

- (1) *For a glueing to exist, it is necessary and sufficient that $f'_*[A] = f''^*[B] \in \text{Ext}_{\mathcal{A}}^1(A'', B')$.*
- (2) *For a glueing to be unique, it is necessary and sufficient that $\text{Hom}_{\mathcal{A}}(A'', B') = 0$.*
- (3) *Suppose that \mathcal{A} is the heart of a triangulated category \mathcal{T} provided with a t -structure. Then Condition (1) is equivalent to the following: the diagram*

$$\begin{array}{ccc} A'' & \xrightarrow{[A]} & A'[1] \\ f'' \downarrow & & f' \downarrow \\ B'' & \xrightarrow{[B]} & B'[1] \end{array}$$

commutes in \mathcal{T} .

Proof. (1). The condition is clearly necessary, and it is sufficient since by definition, two extension classes are equal if there exists an isomorphism between the corresponding extensions.

(2) is obvious since a difference of two glueings is given by a map from A'' to B' .

(3) is clear by Axiom TR3 of triangulated categories. \square

E.2. Extensions panachées. Let \mathcal{A} still be an abelian category, and consider now a filtered object

$$0 \subseteq A_{-2} \subseteq A_{\leq -1} \subseteq A.$$

We write $A_{-1} = A_{\leq -1}/A_{-2}$, $A_0 = A/A_{\leq -1}$, $A_{>-2} = A/A_{-2}$, so that we have a commutative diagram of exact sequences

$$\begin{array}{ccccccc}
 & & 0 & & 0 & & \\
 & & \downarrow & & \downarrow & & \\
 0 & \longrightarrow & A_{-2} & \xrightarrow{j} & A_{\leq -1} & \xrightarrow{\pi} & A_{-1} \longrightarrow 0 \\
 & & \parallel & & \tilde{i} \downarrow & & i \downarrow \\
 \text{(E.1)} \quad 0 & \longrightarrow & A_{-2} & \xrightarrow{\tilde{j}} & A & \xrightarrow{\tilde{\pi}} & A_{>-2} \longrightarrow 0 \\
 & & & & \tilde{\omega} \downarrow & & \omega \downarrow \\
 & & & & A_0 & \xlongequal{\quad} & A_0 \\
 & & & & \downarrow & & \downarrow \\
 & & & & 0 & & 0.
 \end{array}$$

From the viewpoint of [SGA7, Exp. IX, §9.3], this displays A as an *extension panachée*¹⁷ of $A_{>-2}$ by $A_{\leq -1}$. We shall call the datum of this diagram minus A a *panachage datum* (donnée de panachage). There is an obvious notion of morphism of panachage data.

The following is a variant of [14, §1].

E.2.1. Lemma. *Given a panachage datum as above,*

- (1) *An extension panachée exists if and only if the Yoneda product of the extensions $A_{\leq -1}$ and $A_{>-2}$ is 0.*
- (2) *An extension panachée of $A_{>-2}$ by $A_{\leq -1}$ corresponds up to isomorphism to:*
 - (i) *an extension class $\alpha \in \text{Ext}_{\mathcal{A}}^1(A_0, A_{\leq -1})$;*
 - (ii) *an extension class $\tilde{\alpha} \in \text{Ext}_{\mathcal{A}}^1(A_{>-2}, A_{-2})$;*
 - such that*
 - (iii) *$\pi_*\alpha = [A_{>-2}]$, $i^*\tilde{\alpha} = [A_{\leq -1}]$.*
- (3) *Let A, A' be two extensions panachées, with classes $(\alpha, \tilde{\alpha}), (\alpha', \tilde{\alpha}')$ as in (1). Then $\alpha - \alpha'$ and $\tilde{\alpha} - \tilde{\alpha}'$ come from classes $\gamma, \delta \in \text{Ext}_{\mathcal{A}}^1(A_0, A_{-2})$, well-defined modulo the images of $\text{Hom}_{\mathcal{A}}(A_0, A_{-1})$ and $\text{Hom}_{\mathcal{A}}(A_{-1}, A_{-2})$ respectively. Moreover,*

$$\gamma = \delta \in \text{Coker} \left(\text{Hom}_{\mathcal{A}}(A_0, A_{-1}) \oplus \text{Hom}_{\mathcal{A}}(A_{-1}, A_{-2}) \rightarrow \text{Ext}_{\mathcal{A}}^1(A_0, A_{-2}) \right).$$

Proof. (1) is [SGA7, Exp. IX, 9.3.8 c)]. (2) is obvious as well as the existence of γ and δ in (3). The equality can be proven as in [14, lemme

¹⁷L. Breen suggested *blended extension* for an English translation: we learned this from D. Bertrand.

2]. Another way to prove it is to consider the diagram

$$\begin{array}{ccccc}
& & \text{Hom}_{\mathcal{A}}(A_{-1}, A_{-2}) & & \\
& & \downarrow & & \\
\text{Hom}_{\mathcal{A}}(A_0, A_{-1}) & \longrightarrow & \text{Ext}_{\mathcal{A}}^1(A_0, A_{-2}) & \xrightarrow{j^*} & \text{Ext}_{\mathcal{A}}^1(A_0, A_{\leq -1}) \\
& & \varpi^* \downarrow & & \varpi^* \downarrow \\
& & \text{Ext}_{\mathcal{A}}^1(A_{> -2}, A_{-2}) & \xrightarrow{j^*} & \text{Ext}_{\mathcal{A}}^1(A_{> -2}, A_{\leq -1})
\end{array}$$

and to apply a variant of [15, Lemma 2.8]. \square

E.3. The case of 3-step filtrations. Let \mathcal{A} still be an abelian category, and consider now two filtered objects

$$\begin{aligned}
0 &\subseteq A_{-2} \subseteq A_{\leq -1} \subseteq A \\
0 &\subseteq B_{-2} \subseteq B_{\leq -1} \subseteq B.
\end{aligned}$$

E.3.1. Definition. Suppose given morphisms $f_i : A_i \rightarrow B_i$. A *glueing* of the f_i is a filtered morphism $f : A \rightarrow B$ inducing the f_i on the associated graded. A *partial glueing* is a morphism of panachage data inducing the f_i .

We want to find a condition for the f_i to glue. We shall make the following simplifying hypothesis:

E.3.2. Hypothesis. $\text{Hom}_{\mathcal{A}}(A_i, B_j) = 0$ for $i > j$.

E.3.3. Proposition.

- (1) *If a glueing exists, it is unique.*
- (2) *A necessary condition is that Condition (1) or (2) of Lemma E.1.1 is satisfied for the pairs (f_{-2}, f_{-1}) and (f_{-1}, f_0) , relative to the extensions $A_{\leq -1}, B_{\leq -1}$ and $A_{> -2}, B_{> -2}$.*
- (3) *Suppose the conditions of (2) are satisfied. Then we get a (unique) partial glueing.*
- (4) *Given a morphism of panachage data $(f_{-2}, f_{-1}, f_0, f_{\leq -1}, f_{> -2})$, we have two obstructions in $\text{Ext}_{\mathcal{A}}^1(A_0, B_{-2})$ to the existence of a glueing, given respectively (with obvious notation) by $(f_{\leq -1})_*[A] - f_0^*[B]$ and $(f_{-2})_*[A] - f_{> -2}^*[B]$.*
- (5) *The glueing exists if and only if either of the two obstructions of (4) vanishes.*
- (6) *If all f_n are isomorphisms, so is f .*

Proof. (1) is easy, (2) is clear and (3) follows from Lemma E.1.1.

For (4), we have to check that these extension classes, which are a priori respectively in $\text{Ext}_{\mathcal{A}}^1(A_0, B_{\leq -1})$ and in $\text{Ext}_{\mathcal{A}}^1(A_{> -2}, B_{-2})$, define

unique elements of $\text{Ext}_{\mathcal{A}}^1(A_0, B_{-2})$. For the existence, it suffices to see that their images in $\text{Ext}_{\mathcal{A}}^1(A_{-1}, B_{-2})$ are 0, which follows from the conditions of (2), while the uniqueness follows from E.3.2.

(5) follows from Lemma E.1.1. (6) is clear. \square

E.3.4. Remark. It is likely that the two obstructions of Proposition E.3.3 (4) are opposite, but we don't have a good proof.

E.4. Right adjoints. We let \mathcal{A} be an abelian category. Consider the situation

$$(E.2) \quad 0 \rightarrow \mathcal{A}' \xrightarrow{i} \mathcal{A} \xrightarrow{\pi} \mathcal{A}'' \rightarrow 0$$

where $\mathcal{A}' \subseteq \mathcal{A}$ is a thick subcategory and π is the corresponding localisation functor. Thus \mathcal{A}'' is the Serre quotient of \mathcal{A} by \mathcal{A}' . Note that i and π are exact.

Let $A \in \mathcal{A}$. Recall that, by definition, the right adjoint p of i is *defined at* A if the functor

$$\mathcal{A}' \ni B \mapsto \text{Hom}_{\mathcal{A}}(iB, A)$$

is representable. A representative object is then unique up to unique isomorphism: we write it pA and call it the *value* of p at A . We then have a canonical “counit” map

$$ipA \rightarrow A$$

given by the universal property of pA .

Define the following full subcategories of \mathcal{A} :

$$(E.3) \quad \begin{aligned} \mathcal{A}_1 &= \{A \in \mathcal{A} \mid p \text{ is defined at } A\} \\ \mathcal{B} &= \{B \in \mathcal{A}_1 \mid pB = 0\} \\ \mathcal{A}_2 &= \{A \in \mathcal{A} \mid \exists \text{ exact sequence } 0 \rightarrow iA' \rightarrow A \rightarrow A'' \rightarrow 0 \\ &\quad \text{with } A' \in \mathcal{A}', A'' \in \mathcal{B}\} \\ \mathcal{B}' &= \{B \in \mathcal{B} \mid \text{Hom}_{\mathcal{A}}(B, i\mathcal{A}') = 0\} \\ \mathcal{A}_3 &= \{A \in \mathcal{A}_2 \mid A'' \in \mathcal{B}'\}. \end{aligned}$$

E.4.1. Proposition.

- (1) For $A \in \mathcal{A}_1$, the counit map $f : ipA \rightarrow A$ is a monomorphism.
- (2) $\text{Hom}_{\mathcal{A}}(i\mathcal{A}', \mathcal{B}) = 0$. In particular, $i\mathcal{A}' \cap \mathcal{B} = 0$.
- (3) $\mathcal{A}_2 \subseteq \mathcal{A}_1$, with $A' = ipA$ for $A \in \mathcal{A}_2$. In particular, A' is unique and functorial.
- (4) \mathcal{A}_3 is closed under subobjects and quotients (with induced filtrations); in particular, it is abelian, the inclusion functor $\mathcal{A}_3 \hookrightarrow \mathcal{A}$ is exact and morphisms in \mathcal{A}_3 are strictly compatible with the filtrations in the sense of [24, (1.1.5)].

(5) *The restriction of p to \mathcal{A}_3 is exact.*

Proof. (1) Since \mathcal{A}' is thick in \mathcal{A} , $\text{Ker } f = i(K)$ for some $K \in \mathcal{A}'$. Thus we have an exact sequence

$$0 \rightarrow iK \rightarrow ipA \rightarrow A.$$

By adjunction, we get a complex (0-sequence) in \mathcal{A}'

$$K \rightarrow pA \xrightarrow{\cong} pA$$

which shows that $K \rightarrow pA$ is the 0 map. Hence $iK \rightarrow ipA$ is the 0 map, which means that $iK = 0$. Hence f is a monomorphism.

(2) is obvious by adjunction. For “in particular”, if $C \in i\mathcal{A}' \cap \mathcal{B}$, then $1_C \in \text{Hom}_{\mathcal{A}}(C, C) = 0$.

(3) Let $A_1 \in \mathcal{A}'$. Then we have a short exact sequence

$$0 \rightarrow \text{Hom}_{\mathcal{A}}(iA'_1, iA') \rightarrow \text{Hom}_{\mathcal{A}}(iA'_1, A) \rightarrow \text{Hom}_{\mathcal{A}}(iA'_1, A'')$$

in which the last term is 0 by (2).

(4) easily follows from the condition $\text{Hom}_{\mathcal{A}}(\mathcal{B}', i\mathcal{A}') = 0$.

(5) follows from (4). \square

E.4.2. Corollary. *In the situation of (E.2), suppose that i has an everywhere defined right adjoint p . Then the following conditions are equivalent:*

- (i) *p is exact.*
- (ii) *The full subcategory $\text{Ker } p = \{A \mid pA = 0\}$ is stable under quotients and contains A/ipA for any $A \in \mathcal{A}$.*
- (iii) *In (E.3), $\mathcal{A}_3 = \mathcal{A}$.*

Proof. (i) \Rightarrow (ii): Applying p to the exact sequence

$$0 \rightarrow ipA \rightarrow A \rightarrow A/ipA \rightarrow 0$$

we get an exact sequence

$$0 \rightarrow pA \xrightarrow{\cong} pA \rightarrow p(A/ipA) \rightarrow 0$$

which shows that $p(A/ipA) = 0$. Moreover, $\text{Ker } p$ is clearly stable under quotients.

(ii) \Rightarrow (iii): by hypothesis we have $\mathcal{A}_2 = \mathcal{A}$, so we are left to show that $\text{Hom}_{\mathcal{A}}(\text{Ker } p, i\mathcal{A}') = 0$. Let $B \in \text{Ker } p$, $A' \in \mathcal{A}'$, $f : B \rightarrow A'$ and $C = \text{Im } f$. By (ii), $C \in \text{Ker } p$; since i is thick, we also have $C \in \mathcal{A}'$; hence $C = 0$ thanks to Proposition E.4.1 (2).

(iii) \Rightarrow (i): this follows from Proposition E.4.1 (5). \square

E.5. Split exact sequences. We are still in the situation of (E.2).

E.5.1. Proposition.

- (1) *The following conditions are equivalent:*
- (i) *i has an exact right adjoint p .*
 - (ii) *π has an exact right adjoint j and $j(\mathcal{A}'')$ is thick in \mathcal{A} .*
 - (iii) *For any $A \in \mathcal{A}$ there exists $A' \in \mathcal{A}'$ and a monomorphism $i(A') \rightarrow A$ such that*

$$\mathrm{Hom}_{\mathcal{A}}(i(\mathcal{A}'), A/i(A')) = \mathrm{Ext}_{\mathcal{A}}^1(i(\mathcal{A}'), A/i(A')) = 0.$$

- (iv) *Same as (iii), replacing the condition $\mathrm{Ext}_{\mathcal{A}}^1(i(\mathcal{A}'), A/i(A')) = 0$ by $\mathrm{Hom}_{\mathcal{A}}(A/i(A'), i(\mathcal{A}')) = 0$.*
- (2) *If these conditions are verified, then for any $A \in \mathcal{A}$ there exists a unique exact sequence*

$$0 \rightarrow iA' \rightarrow A \rightarrow jA'' \rightarrow 0$$

such that $A' \in \mathcal{A}'$ and $A'' \in \mathcal{A}''$; this sequence is functorial in A . Moreover, we have $\mathrm{Hom}_{\mathcal{A}}(jA'', iA') = 0$ and $\mathrm{Ext}_{\mathcal{A}}^r(iA', jA'') = 0$ for all $r \geq 0$.

- (3) *If \mathcal{A} is semi-simple, the conditions of (1) are verified.*

Proof. (1) (i) \Rightarrow (ii): Let $\mathcal{B} := \mathrm{Ker} p$. By Corollary E.4.2 (ii), we have a functor $q : \mathcal{A} \rightarrow \mathcal{B}$ given by $qA = A/ipA$. The exactness of i and p plus the snake lemma show that q is exact, and q is easily seen to be left adjoint to the inclusion $\mathcal{B} \hookrightarrow \mathcal{A}$. Moreover $q|_{\mathcal{A}'} = 0$, hence q induces a functor $\bar{q} : \mathcal{A}'' \rightarrow \mathcal{B}$.

On the other hand, we have the obvious functor

$$r : \mathcal{B} \rightarrow \mathcal{A} \rightarrow \mathcal{A}''.$$

Let $B \in \mathcal{B}$. By definition, $\bar{q}rB$ is represented by $B/ipB = B$. This provides a natural isomorphism $B \xrightarrow{\sim} \bar{q}rB$.

Let $A \in \mathcal{A}$. The projection $A \rightarrow A/ipA$ induces another natural isomorphism $A \xrightarrow{\sim} r\bar{q}A$, where A is now viewed in \mathcal{A}'' . So \bar{q} and r are quasi-inverse equivalences of categories. Under this equivalence, q becomes π , and the inclusion $\mathcal{B} \hookrightarrow \mathcal{A}$ defines a functor $j : \mathcal{A}'' \rightarrow \mathcal{A}$, which is right adjoint to π . Finally, since $\mathcal{B} = \mathrm{Ker} p$ is thick, $j(\mathcal{A}'')$ is thick.

(ii) \Rightarrow (iii): let $A \in \mathcal{A}$. The functor π kills the kernel and cokernel of the unit map $f : A \rightarrow j\pi A$. By the thickness assumption, $\mathrm{Coker} f \in j(\mathcal{A}'')$ which implies that f is epi; on the other hand, we get $\mathrm{Ker} f \in i(\mathcal{A}')$. Thus we have an exact sequence

$$0 \rightarrow iK \rightarrow A \rightarrow j\pi A \rightarrow 0.$$

If $A' \in \mathcal{A}'$, then

$$\mathrm{Hom}(iA', j\pi A) = \mathrm{Hom}(\pi iA', \pi A) = \mathrm{Hom}(0, \pi A) = 0.$$

On the other hand, let $0 \rightarrow j\pi A \xrightarrow{\alpha} E \rightarrow iA' \rightarrow 0$ be an extension. Since $\pi i = 0$ and π is exact, $\pi(\alpha)$ is an isomorphism. Its inverse $\pi(\alpha)^{-1} : \pi E \rightarrow \pi A$ yields by adjunction a map $\beta : E \rightarrow j\pi A$ and one verifies that β is a retraction of α .

(iii) \implies (i): let $A, B \in \mathcal{A}$, with $A', B' \in \mathcal{A}'$ as in (iii), and let $f : A \rightarrow B$ be a morphism. By assumption, the composition

$$iA' \rightarrow A \xrightarrow{f} B \rightarrow B/iB'$$

is 0, hence f induces a map $A' \rightarrow B'$. This shows that A' is unique up to unique isomorphism (take $f = 1_A$) and is functorial in A . Thus we have a functor $p : \mathcal{A} \rightarrow \mathcal{A}'$ mapping A to A' , and exactly the same argument shows that, for $B \in \mathcal{A}'$, a map $iB \rightarrow A$ induces a map $iB \rightarrow ipA$, hence $B \rightarrow pA$ since i is fully faithful. This yields the claimed adjunction.

It remains to see that p is right exact. For this, we may apply the criterion (ii) of Corollary E.4.2. Let $A \in \mathcal{A}$: then $ip(A/ipA)$ is a subobject of A/ipA , hence is 0 by the hypothesis $\mathrm{Hom}(i(\mathcal{A}'), A/ipA) = 0$. Thus $A/ipA \in \mathrm{Ker} p$. Let now $B \in \mathrm{Ker} p$ and C be a quotient of B so that we have an exact sequence

$$(E.4) \quad 0 \rightarrow A \rightarrow B \rightarrow C \rightarrow 0.$$

Note that $A \in \mathrm{Ker} p$. Pulling back the extension (E.4) via the monomorphism $ipC \rightarrow C$, we get an extension of ipC by A , which is split by hypothesis. A given splitting yields a monomorphism $f : ipC \hookrightarrow B$, which is 0 again by hypothesis, and finally $C \in \mathrm{Ker} p$.

(i) \implies (iv): this follows from (i) \implies (iii) in Corollary E.4.2 and from Proposition E.4.1 (2).

(iv) \implies (i): in (iv), the condition $\mathrm{Hom}_{\mathcal{A}}(i(\mathcal{A}'), A/i(A')) = 0$ implies a fortiori that p is defined at A , with value A' . This condition is then equivalent to $p(A/iA') = 0$, thus (iv) implies Condition (iii) in Corollary E.4.2.

(2) Everything follows from (1), except the vanishing of the higher Ext 's; the claimed exact sequence is given by

$$(E.5) \quad 0 \rightarrow ipA \rightarrow A \rightarrow j\pi A \rightarrow 0.$$

For Ext^r with $r > 0$, we argue by induction on r . This reduces us to show that, for all $r > 0$ and all $A' \in \mathcal{A}'$, the functor

$$\mathcal{A}'' \ni A'' \mapsto \mathrm{Ext}_{\mathcal{A}}^r(iA', jA'')$$

is effaceable.

Let $\mathcal{E} \in \text{Ext}_{\mathcal{A}}^r(iA', jA'')$, represented by the exact sequence

$$0 \rightarrow jA'' \xrightarrow{f} E_1 \rightarrow \cdots \rightarrow E_r \rightarrow iA' \rightarrow 0.$$

Clearly, $f_*\mathcal{E} = 0 \in \text{Ext}_{\mathcal{A}}^r(iA', E_1)$ (to see this, let $F = \text{Coker } f$, and note that \mathcal{E} is the image of $[0 \rightarrow F \rightarrow E_2 \rightarrow \cdots \rightarrow E_r \rightarrow iA' \rightarrow 0] \in \text{Ext}_{\mathcal{A}}^{r-1}(iA', F)$ by the boundary map associated to the exact sequence $0 \rightarrow jA'' \rightarrow E_1 \rightarrow F \rightarrow 0$). A fortiori, \mathcal{E} maps to 0 under the composition

$$\text{Ext}_{\mathcal{A}}^r(iA', jA'') \xrightarrow{f_*} \text{Ext}_{\mathcal{A}}^r(iA', E_1) \rightarrow \text{Ext}_{\mathcal{A}}^r(iA', j\pi E_1).$$

It remains to observe that the composition $jA'' \rightarrow E_1 \rightarrow j\pi E_1$ is a monomorphism: indeed, it is the image of the monomorphism $jA'' \rightarrow E_1$ under the exact functor $j\pi$.

(3) Let I be the set of isomorphism classes of simple objects of \mathcal{A} , and J the subset of I defined by simple objects of \mathcal{A} belonging to \mathcal{A}' . For any $A \in \mathcal{A}$, we have a direct sum decomposition

$$A = A_J \oplus A_{I-J}$$

where A_J (*resp.* A_{I-J}) is the sum of simple subobjects of A whose class belongs to J (*resp.* to $I - J$). Clearly, $A_J \in \mathcal{A}'$, and $\text{Hom}(A_J, A_{I-J}) = 0$. Also, $\text{Ext}^1(A_J, A_{I-J}) = 0$ since \mathcal{A} is semi-simple. Thus Condition (iii) of (1) is satisfied. \square

E.5.2. Definition. In the situation of Proposition E.5.1, we say that the exact sequence (E.2) is *split*.

We shall need the following lemma:

E.5.3. Lemma. *In the situation of Proposition E.5.1, let \mathcal{B} be an additive category and let $E : \mathcal{A}'' \rightarrow \mathcal{B}$, $F : \mathcal{A}' \rightarrow \mathcal{B}$ be two additive functors. Then a natural transformation $E\pi \Rightarrow Fp$ on \mathcal{A} is equivalent to a bivariant transformation*

$$\text{Ext}_{\mathcal{A}}^1(jA'', iA') \rightarrow \text{Hom}_{\mathcal{B}}(E(A''), F(A'))$$

on $\mathcal{A}'' \times \mathcal{A}'$.

Proof. Let $u_A : E\pi A \rightarrow FpA$ be a natural transformation. For $(A', A'') \in \mathcal{A}' \times \mathcal{A}''$ and two extensions A_1, A_2 of jA'' by iA' in \mathcal{A} , the Baer sum $A_1 \boxplus A_2$ may be obtained as $\Delta^* \Sigma_*(A_1 \oplus A_2)$, where $\Delta : jA'' \rightarrow jA'' \oplus jA''$ is the diagonal map and $\Sigma : iA' \oplus iA' \rightarrow iA'$ is the sum map. Since $u_{A_1 \oplus A_2} = u_{A_1} \oplus u_{A_2}$, this implies that

$$u_{A_1 \boxplus A_2} = u_{A_1} + u_{A_2}$$

so that u induces a homomorphism as in the lemma. The converse will not be used and is left to the interested reader. \square

E.6. More adjoints. Assume that (E.2) is split. Let us now consider the conditions

- (b) j has a right adjoint π' .
- (#) i has a left adjoint p' .

Recall that, under (b), we get a canonical natural transformation $\alpha : \pi' \Rightarrow \pi$ from the composition $j\pi'A \rightarrow A \rightarrow j\pi A$ for any $A \in \mathcal{A}$, and the full faithfulness of j . Similarly, under (#), we get a canonical natural transformation $\beta : p \Rightarrow p'$.

E.6.1. Proposition. *We have (b) \iff (#). If this holds, then $\pi'i = 0$, $p'j = 0$, the natural transformation α is a monomorphism and the natural transformation β is an epimorphism.*

If \mathcal{A}'' is Noetherian (resp. \mathcal{A}' is Artinian), then (b) (resp. (#)) holds.

Proof. Suppose that π' exists. Since $\text{Hom}_{\mathcal{A}}(j\mathcal{A}'', i\mathcal{A}') = 0$ (Proposition E.5.1 (2)), we have $\text{Hom}_{\mathcal{A}''}(\mathcal{A}'', \pi'i\mathcal{A}') = 0$, i.e. $\pi'i = 0$. Note that π' is left exact, as a right adjoint. Let now $A \in \mathcal{A}$. Applying π' to the exact sequence (E.5), we get an exact sequence

$$0 \rightarrow 0 = \pi'ipA \rightarrow \pi'A \xrightarrow{\alpha_A} \pi A.$$

This shows that α_A is injective. Dually, if p' exists then $p'j = 0$ and β is an epimorphism.

Suppose that π' exists. Let $A \in \mathcal{A}$, $A' \in \mathcal{A}'$: the exact sequence

$$0 \rightarrow \text{Hom}_{\mathcal{A}}(A/j\pi'A, iA') \rightarrow \text{Hom}_{\mathcal{A}}(A, iA') \rightarrow \text{Hom}_{\mathcal{A}}(j\pi'A, iA') = 0$$

shows that $A \mapsto A/j\pi'A$ defines a left adjoint p' to i . We get the reverse implication by the dual reasoning.

It remains to prove the last assertions. Let $A \in \mathcal{A}$. For a subobject B of πA , consider the pull-back $A' \xrightarrow{\varphi} jB$ of the map $A \rightarrow j\pi A$. We define $\pi'A$ as the largest B such that φ is split: clearly, $\pi'A$ is functorial in A . Moreover, a splitting of φ is unique (still by Proposition E.5.1 (2)), hence defines a natural transformation

$$\varepsilon_A : j\pi'A \rightarrow A' \rightarrow A.$$

On the other hand, $\pi'j$ is clearly the identity functor; define then for $A'' \in \mathcal{A}''$ a unit map $\eta_{A''} : A'' \rightarrow \pi'jA''$ as the identity map. The adjunction identities are readily checked. The case of (#) is dealt with dually. \square

E.7. Filtrations.

E.7.1. Definition. Let \mathcal{A} be an abelian category.

a) A *filtration* on \mathcal{A} is a sequence of thick subcategories

$$(E.6) \quad \dots \xrightarrow{\iota_{n-1}} \mathcal{A}_{\leq n} \xrightarrow{\iota_n} \mathcal{A}_{\leq n+1} \xrightarrow{\iota_{n+1}} \dots \rightarrow \mathcal{A}.$$

It is *separated* if $\bigcap \mathcal{A}_{\leq n} = \{0\}$ and *exhaustive* if $\mathcal{A} = \bigcup \mathcal{A}_{\leq n}$.

b) A filtration on \mathcal{A} is *split* if all ι_n have exact right adjoints.

E.7.2. Definition. A filtration on \mathcal{A} is a *weight filtration* if it is separated, exhaustive and split.

Let \mathcal{A} be provided with a split filtration (Definition E.7.1). Define

$$\mathcal{A}_n = \mathcal{A}_{\leq n} / \mathcal{A}_{\leq n-1}, \quad \mathcal{A}_{n,n+1} = \mathcal{A}_{\leq n+1} / \mathcal{A}_{\leq n-1}$$

so that we have exact sequences

$$(E.7) \quad 0 \rightarrow \mathcal{A}_n \xrightarrow{i_n} \mathcal{A}_{n,n+1} \xrightarrow{\pi_n} \mathcal{A}_{n+1} \rightarrow 0$$

where i_n is induced by ι_n . Let us write j_n for the right adjoint of the localisation functor $\mathcal{A}_{\leq n} \rightarrow \mathcal{A}_{\leq n} / \mathcal{A}_{\leq n-1}$ (Proposition E.5.1). By abuse of notation, we shall identify \mathcal{A}_n with its thick image in \mathcal{A} via j_n (*ibid.*).

E.7.3. Lemma. *If ι_n (E.6) has a right (resp. left) adjoint, so does i_n (E.7). If one is exact, so is the other.*

Proof. We have a diagram of exact sequences of abelian categories:

$$\begin{array}{ccccccc} & & \mathcal{A}_{\leq n-1} & \xlongequal{\quad} & \mathcal{A}_{\leq n-1} & & \\ & & \downarrow \iota_{n-1} & & \downarrow \iota_n \iota_{n-1} & & \\ 0 & \longrightarrow & \mathcal{A}_{\leq n} & \xrightarrow{\iota_n} & \mathcal{A}_{\leq n+1} & \xrightarrow{\tilde{\pi}_n} & \mathcal{A}_{n+1} \longrightarrow 0 \\ & & \downarrow \tilde{\pi}_{n-1} & & \downarrow q_{n-1} & & \downarrow \parallel \\ 0 & \longrightarrow & \mathcal{A}_n & \xrightarrow{i_n} & \mathcal{A}_{n,n+1} & \xrightarrow{\pi_n} & \mathcal{A}_{n+1} \longrightarrow 0. \end{array}$$

Let ϖ_n be a right adjoint to ι_n . Since ι_n is fully faithful, the unit map $\eta_n : Id_{\mathcal{A}_{\leq n}} \Rightarrow \varpi_n \iota_n$ is an isomorphism. Hence $\varpi_n(\iota_{n-1} \mathcal{A}_{\leq n-1}) \subseteq \iota_{n-1} \mathcal{A}_{\leq n-1}$ and ϖ_n induces a functor $p_n : \mathcal{A}_{n,n+1} \rightarrow \mathcal{A}_n$. Let $\varepsilon_n : \iota_n \varpi_n \Rightarrow Id_{\mathcal{A}_{\leq n+1}}$ be the counit map of the adjunction: then η_n and ε_n induce natural transformations $p_{n-1} * \eta_n : Id_{\mathcal{A}_n} \Rightarrow p_n i_n$ and $q_{n-1} * \varepsilon_n : i_n p_n \Rightarrow Id_{\mathcal{A}_{n,n+1}}$. Since η_n and ε_n verify the identities of [59, p. 82, Th. 1 (8)], these identities are preserved when applying p_{n-1} and q_{n-1} , hence $p_{n-1} * \eta_n$ and $q_{n-1} * \varepsilon_n$ define an adjunction between i_n and p_n by *ibid.*, p. 83, Th. 2 (v). (Alternately, one can check directly that p_n is right adjoint to i_n .) The assertion on exactness is true because the functor q_{n-1} is exact. The reasoning is the same for a left adjoint. \square

E.7.4. Proposition. *Suppose that \mathcal{A} is provided with an exhaustive split filtration as in Definition E.7.1 a), b). Then:*

- (1) Every object $A \in \mathcal{A}$ is provided with a unique filtration

$$\cdots \subseteq A_{\leq n} \subseteq A_{\leq n+1} \subseteq \cdots \subseteq A$$

with $A_{\leq n} \in \mathcal{A}_{\leq n}$ and $A_n := A_{\leq n}/A_{\leq n-1} \in \mathcal{A}_n$. One has $A_{\leq n} = A$ for n large enough.

- (2) The functors $A \mapsto A_{\leq n}$ and $A \mapsto A_n$ are exact, as well as $A \mapsto A_{\geq n} := A/A_{\leq n-1}$.
 (3) If moreover the filtration is separated, the $A \mapsto A_n$ form a set of conservative functors.
 (4) We have $\mathrm{Hom}_{\mathcal{A}}(\mathcal{A}_n, \mathcal{A}_{\leq n-1}) = 0$ and $\mathrm{Ext}_{\mathcal{A}}^i(\mathcal{A}_{\leq n-1}, \mathcal{A}_n) = 0$ for all $i \geq 0$ for all $n \in \mathbb{Z}$ and $i \geq 0$.

Proof. (1) Since $\mathcal{A} = \bigcup \mathcal{A}_{\leq n}$, there exists n_0 such that $A \in \mathcal{A}_{\leq n_0}$. For $n < n_0$, write I_n for the inclusion $\mathcal{A}_{\leq n} \rightarrow \mathcal{A}_{\leq n_0}$ and P_n for its right adjoint, which exists and is exact as a composition of the right adjoints of the ι_r for $n \leq r < n_0$. Define

$$A_{\leq n} = I_n P_n A.$$

Since $A = \iota_{n_0} \varpi_{n_0} A$, $A_{\leq n}$ does not depend on the choice of n_0 , and we have a filtration of A as in Proposition E.7.4. Clearly, $A_{\leq n} \in \mathcal{A}_{\leq n}$ and $A_{\leq n_0} = A$. The fact that $A_n \in \mathcal{A}_n$ follows from Proposition E.5.1 (2), which also shows the uniqueness of the filtration.

(2) This still follows from Proposition E.5.1 (2).

(3) Let $f : A \rightarrow B$ be such that $f_n : A_n \rightarrow B_n$ is an isomorphism for all $n \in \mathbb{Z}$. Let $K = \mathrm{Ker} f$ and $C = \mathrm{Coker} f$. By exactness, $K_n = C_n = 0$ for all n . Thus $K, C \in \bigcap \mathcal{A}_{\leq n} = 0$ and f is an isomorphism.

(4) This follows from Proposition E.5.1 (2) again. \square

E.7.5. Lemma. Assume given a weight filtration $\mathcal{A}_{\leq n}$ on \mathcal{A} and assume that the categories \mathcal{A}_n are semi-simple. Let $A \in \mathcal{A}_m$ and $B \in \mathcal{A}_n$. Then, $\mathrm{Ext}_{\mathcal{A}}^i(A, B) = 0$ if $i > m - n$.

Proof. By induction on i . For $i = 0$, it means that $\mathrm{Hom}_{\mathcal{A}}(A, B) = 0$ if $m < n$, which is true by E.7.4 (4). For $i = 1$, it means that $\mathrm{Ext}_{\mathcal{A}}^1(A, B) = 0$ if $m \leq n$: for $m < n$ this is still E.7.4 (4) and for $m = n$ it comes from the semi-simplicity of \mathcal{A}_n and its thickness in \mathcal{A} (E.5.1 (1) (ii)).

Suppose now $i > 1$ and $m - n < i$. Let $\alpha \in \mathrm{Ext}_{\mathcal{A}}^i(A, B)$. We may write α as a Yoneda product

$$\alpha = \gamma \beta$$

with $\beta \in \mathrm{Ext}_{\mathcal{A}}^1(A, C)$ and $\gamma \in \mathrm{Ext}_{\mathcal{A}}^{i-1}(C, B)$ for some $C \in \mathcal{A}$. By induction, the map

$$\mathrm{Ext}_{\mathcal{A}}^{i-1}(C_{\geq n+i-1}, B) \rightarrow \mathrm{Ext}_{\mathcal{A}}^{i-1}(C, B)$$

is surjective, hence we may assume $C \in \mathcal{A}_{\geq n+i-1}$. But then $\mathrm{Ext}_{\mathcal{A}}^1(A, C) = 0$ and we are done. \square

E.7.6. Definition. Let \mathcal{A} be provided with an exhaustive split filtration. Let $A \in \mathcal{A}$. We say that A has *finite length* if $A_{\leq n} = 0$ for n small enough. The length of A is then the integer

$$\ell(A) = n - m$$

where n is the smallest integer such that $A_{\leq n} = A$ and m is the largest integer such that $A_{\leq m} = 0$.

E.7.7. Lemma. Let \mathcal{A} be provided with a weight filtration $(\mathcal{A}_{\leq n}, \iota_n)$. Let $A, B \in \mathcal{A}$, with B of finite length. Let $f : A \rightarrow B$ be a morphism such that $f_n : A_n \rightarrow B_n$ is 0 for all n . Then $f = 0$.

Proof. The assumption implies that $f(A_{\leq n}) \subseteq B_{\leq n-1}$ for all $n \in \mathbb{Z}$. Hence f induces morphisms $f_n^{(1)} : A_n \rightarrow B_{n-1}$, which are 0 by Proposition E.7.4 (4). Inductively on k , we get $f(A_{\leq n}) \subseteq B_{\leq n-k}$ for all $n \in \mathbb{Z}$ and all $k \geq 0$. For n large enough we have $A_{\leq n} = A$ and for k large enough we have $B_{\leq n-k} = 0$, hence $f = 0$. \square

E.7.8. Remark. If \mathcal{A} is provided with a weight filtration such that every object has finite length, then the functors $A \mapsto A_{\leq n}$ of Proposition E.7.4 (1) verify Jannsen's conditions in [46, p. 83, Def. 6.3 a)]. Conversely, let as in *loc. cit.* $(W_n)_{n \in \mathbb{Z}}$ be an increasing sequence of exact subfunctors of $\mathrm{Id}_{\mathcal{A}}$ such that, for all $A \in \mathcal{A}$, one has $W_n A = 0$ for $n \ll 0$ and $W_n A = A$ for $n \gg 0$. Define $\mathcal{A}_{\leq n}$ as the full subcategory of \mathcal{A} consisting of objects A such that $W_n A = A$. This filtration is clearly separated and exhaustive in the sense of Definition E.7.1. Moreover, for all $n \in \mathbb{Z}$, $\iota_n : \mathcal{A}_{\leq n} \hookrightarrow \mathcal{A}_{\leq n+1}$ has the exact right adjoint W_n , so our filtration is also split. Summarising, the datum of a weight filtration as in [46, p. 83, Def. 6.3 a)] is equivalent to that of a weight filtration in the sense of Definition E.7.2 for which every object has finite length.

In view of this remark, the following is an abstract version of [46, p. 87, Ex. 6.8]:

E.7.9. Proposition. Let \mathcal{A} be provided with a filtration $(\mathcal{A}_{\leq n})_{n \in \mathbb{Z}}$. For each $n \in \mathbb{Z}$, let \mathcal{A}'_n be a full subcategory of $\mathcal{A}_{\leq n}$ such that

- (i) For all $n \in \mathbb{Z}$, \mathcal{A}'_n is abelian.
- (ii) For $m \neq n$, $\mathrm{Hom}_{\mathcal{A}}(\mathcal{A}'_m, \mathcal{A}'_n) = 0$.

Let \mathcal{A}' be the full subcategory of \mathcal{A} consisting of objects A admitting a finite increasing filtration $(A_{\leq n})$ with

- (i) $A_{\leq n} = 0$ for $n \ll 0$.

- (ii) $A_{\leq n} = A$ for $n \gg 0$.
- (iii) $A_{\leq n}/A_{\leq n-1} \in \mathcal{A}'_n$ for all $n \in \mathbb{Z}$.

Then \mathcal{A}' is an abelian subcategory of \mathcal{A} , closed under subobjects and quotients. The above filtration is unique for any $A \in \mathcal{A}'$, and any morphism is strict. If $\mathcal{A}'_{\leq n} := \mathcal{A}' \cap \mathcal{A}_{\leq n}$, then the filtration $\mathcal{A}'_{\leq n}$ is a weight filtration and any object of \mathcal{A}' has finite length.

Proof. In principle this follows from Proposition E.4.1 by induction, but it would be tedious to write up (the induction would have to bear on the length of the filtration of an object). Instead, we observe that the proof in [46, p. 88, Lemma 6.8.1] shows that \mathcal{A}' is provided with a weight filtration in the sense of [46, p. 83, Def. 6.3 a)], and we apply Remark E.7.8. \square

II. MORPHISMS OF FILTERED CATEGORIES

E.8. The case of a 2-step filtration.

E.8.1. Proposition. *Let $0 \rightarrow \mathcal{A}' \xrightarrow{i} \mathcal{A} \xrightarrow{\pi} \mathcal{A}'' \rightarrow 0$ and $0 \rightarrow \mathcal{B}' \xrightarrow{i'} \mathcal{B} \xrightarrow{\pi'} \mathcal{B}'' \rightarrow 0$ be as in (E.2), and consider a naturally commutative diagram of exact functors*

$$\begin{array}{ccccccc}
 0 & \longrightarrow & \mathcal{A}' & \xrightarrow{i} & \mathcal{A} & \xrightarrow{\pi} & \mathcal{A}'' \longrightarrow 0 \\
 & & R' \downarrow & & R \downarrow & & R'' \downarrow \\
 0 & \longrightarrow & \mathcal{B}' & \xrightarrow{i'} & \mathcal{B} & \xrightarrow{\pi'} & \mathcal{B}'' \longrightarrow 0.
 \end{array}$$

Suppose that the two rows are split in the sense of Definition E.5.2; we use the notation (p, j) (resp. (p', j')) for the corresponding adjoints. Then the following are equivalent:

- (i) The natural “base change” transformation $R'p \Rightarrow p'R$ of §15.1 is a natural isomorphism.
- (ii) $Rj \Rightarrow j'R''$ is a natural isomorphism.
- (iii) $R(j\mathcal{A}'') \subseteq j'\mathcal{B}''$.

Proof. Let $A \in \mathcal{A}$. We have a commutative diagram of exact sequences (using (E.5) and the base change morphisms):

$$\begin{array}{ccccccc}
 0 & \longrightarrow & R(ipA) & \longrightarrow & R(A) & \longrightarrow & R(j\pi A) \longrightarrow 0 \\
 & & \downarrow & & \parallel \downarrow & & \downarrow \\
 0 & \longrightarrow & i'p'R(A) & \longrightarrow & R(A) & \longrightarrow & j'\pi'R(A) \longrightarrow 0.
 \end{array}$$

Thus the left vertical map is an isomorphism if and only if the right vertical map is one, which shows that (i) \iff (ii). If this is the case,

then $R(j\pi A) \in j'(\mathcal{B}')$, hence (iii). Conversely, if (iii) holds, then all vertical maps must be isomorphisms by the uniqueness of (E.5). \square

E.9. The general case.

E.9.1. Definition. Let \mathcal{A} and \mathcal{B} two filtered abelian categories $(\mathcal{A}_{\leq n}, \iota_n)$, $(\mathcal{B}_{\leq n}, \iota'_n)$ as in Definition E.7.1. Let $R : \mathcal{A} \rightarrow \mathcal{B}$ be an additive functor.

- (1) We say that R *respects the filtration* if it is exact and $R(\mathcal{A}_{\leq n}) \subseteq \mathcal{B}_{\leq n}$.
- (2) Suppose that the two filtrations are split. We say that R *respects the splittings* if, moreover, $R(\mathcal{A}_n) \subseteq \mathcal{B}_n$ for all $n \in \mathbb{Z}$.
- (3) We denote by $R_{\leq n} : \mathcal{A}_{\leq n} \rightarrow \mathcal{B}_{\leq n}$ the restriction of R to $\mathcal{A}_{\leq n}$ and by $R_n : \mathcal{A}_n \rightarrow \mathcal{B}_n$ its restriction to \mathcal{A}_n , if applicable.

E.9.2. Lemma. Let $\mathcal{A}, \mathcal{B}, R$ be as in Definition E.9.1. Suppose that the filtrations of \mathcal{A} and \mathcal{B} are exhaustive and split. Then:

- (1) If every object of \mathcal{A} has finite length in the sense of Definition E.7.6, then R respects the splittings provided $R(\mathcal{A}_n) \subseteq \mathcal{B}_n$ for all n .
- (2) If R respects the splittings, then for all $n \in \mathbb{Z}$, $R_{\leq n}$ and R_n are exact.
- (3) Let $A \in \mathcal{A}$. Then $R(A_{\leq n}) = R(A)_{\leq n}$ and $R(A_n) \xrightarrow{\sim} R(A)_n$ for any $n \in \mathbb{Z}$.

Proof. (1) Let $A \in \mathcal{A}$. We have to show that, if $A \in \mathcal{A}_{\leq n}$, then $R(A) \in \mathcal{B}_{\leq n}$. We argue by induction on the length $\ell(A)$ of the weight filtration on A . If $\ell(A) = 0$, then $A = 0$ this is clear. If $\ell(A) > 0$, we may assume n minimal. Then $\ell(A_{\leq n-1}) = \ell(A) - 1$, hence $R(A_{\leq n-1}) \in \mathcal{B}_{\leq n-1}$, and the exact sequence

$$0 \rightarrow R(A_{\leq n-1}) \rightarrow R(A) \rightarrow R(A_n) \rightarrow 0$$

with $R(A_n) \in \mathcal{B}_n$ shows that $R(A) \in \mathcal{B}_{\leq n}$.

(2) This follows from the exactness of the inclusions ι_n and j_n , and from the faithful exactness of the ι'_n and j'_n .

(3) Let $A \in \mathcal{A}_{\leq n_0}$; then $R(A) \in \mathcal{B}_{\leq n_0}$. For $n < n_0$, let I'_n be the inclusion $\mathcal{B}_{\leq n} \rightarrow \mathcal{B}_{n_0}$ and P'_n its exact right adjoint. We have

$$R(A_{\leq n}) = I'_n P'_n R(A_{\leq n}) \subseteq I'_n P'_n R(A) = R(A)_{\leq n}.$$

For equality, consider the commutative diagram of exact sequences

$$\begin{array}{ccccccc} 0 & \longrightarrow & R(A_{\leq n}) & \longrightarrow & R(A) & \longrightarrow & R(A_{\geq n+1}) \longrightarrow 0 \\ & & \downarrow a & & \downarrow \parallel & & \downarrow b \\ 0 & \longrightarrow & R(A)_{\leq n} & \longrightarrow & R(A) & \longrightarrow & R(A)_{\geq n+1} \longrightarrow 0. \end{array}$$

(see Proposition E.7.4 (2) for $A_{\geq n+1}$). Let $\mathcal{A}_{[n,n_0]}$ be the image of $\mathcal{A}_{\leq n_0}/\mathcal{A}_{\leq n}$ by the exact right adjoint of the localisation functor, and similarly for \mathcal{B} (see Proposition E.5.1 (1)). Note that $R(\mathcal{A}_{[n,n_0]}) \subseteq \mathcal{B}_{[n,n_0]}$ because $R(\mathcal{A}_i) \subseteq \mathcal{B}_i$ for all $i > n$. Hence $R(A_{>n}) \in \mathcal{B}_{[n,n_0]}$. The snake lemma shows that $\text{Ker } b \xrightarrow{\sim} \text{Coker } a \in \mathcal{B}_{[n,n_0]} \cap \mathcal{B}_{\leq n} = 0$.

The last claim follows. \square

III. GLUEING NATURAL TRANSFORMATIONS

E.10. The case of a 2-step filtration. Let $0 \rightarrow \mathcal{A}' \xrightarrow{i} \mathcal{A} \xrightarrow{\pi} \mathcal{A}'' \rightarrow 0$ and $0 \rightarrow \mathcal{B}' \xrightarrow{i'} \mathcal{B} \xrightarrow{\pi'} \mathcal{B}'' \rightarrow 0$ be as in (E.2), and consider two naturally commutative diagrams of exact functors

$$\begin{array}{ccccccc} 0 & \longrightarrow & \mathcal{A}' & \xrightarrow{i} & \mathcal{A} & \xrightarrow{\pi} & \mathcal{A}'' \longrightarrow 0 \\ & & R'_n \downarrow & & R_n \downarrow & & R''_n \downarrow \\ 0 & \longrightarrow & \mathcal{B}' & \xrightarrow{i'} & \mathcal{B} & \xrightarrow{\pi'} & \mathcal{B}'' \longrightarrow 0 \end{array}$$

for $n = 1, 2$. We assume that the conditions of Proposition E.8.1 are satisfied for $n = 1, 2$. We also assume given two natural transformations $u' : R'_1 \Rightarrow R'_2$ and $u'' : R''_1 \Rightarrow R''_2$.

E.10.1. Definition. A *glueing* of u' and u'' is a natural transformation $u : R_1 \Rightarrow R_2$ such that, for any $A \in \mathcal{A}$, the diagram

$$(E.8) \quad \begin{array}{ccccccc} 0 & \longrightarrow & i'R'_1(pA) & \longrightarrow & R_1(A) & \longrightarrow & j'R''_1(\pi A) \longrightarrow 0 \\ & & i'(u'_{pA}) \downarrow & & u_A \downarrow & & j'(u''_{\pi A}) \downarrow \\ 0 & \longrightarrow & i'R'_2(pA) & \longrightarrow & R_2(A) & \longrightarrow & j'R''_2(\pi A) \longrightarrow 0 \end{array}$$

commutes.

E.10.2. Theorem.

- (1) *For the existence of u , the following condition is necessary: for any $(A', A'') \in \mathcal{A}' \times \mathcal{A}''$, the diagram*

$$\begin{array}{ccc} \text{Ext}_{\mathcal{A}}^i(j'A'', i'A') & \xrightarrow{\overline{R}_1} & \text{Ext}_{\mathcal{B}}^i(j'R''_1 A'', i'R'_1 A') \\ \overline{R}_2 \downarrow & & i'(u'_{A'})_* \downarrow \\ \text{Ext}_{\mathcal{B}}^i(j'R''_2 A'', i'R'_2 A') & \xrightarrow{j'(u''_{A'})^*} & \text{Ext}_{\mathcal{B}}^i(j'R''_1 A'', i'R'_2 A') \end{array}$$

commutes for $i = 0, 1$, where \overline{R}_n denotes the composition of R_n with suitable natural isomorphisms.

We may view this obstruction as a bivariant natural transformation

$$i'(u'_A)_* \overline{R}_1 - j'(u''_{A''})^* \overline{R}_2 : \text{Ext}_{\mathcal{A}}^i(jA'', iA') \rightarrow \text{Ext}_{\mathcal{B}}^i(j'R_1''A'', i'R_2'A').$$

- (2) Suppose that \mathcal{B} is the heart of a t -structure on a triangulated category \mathcal{T} . Then the condition of (1) for $i = 1$ is equivalent to the following: for any $(A', A'') \in \mathcal{A}' \times \mathcal{A}''$, the diagram

$$\begin{array}{ccc} j'R_1''(\pi A) & \xrightarrow{[R_1(A)]} & i'R_1(pA)[1] \\ j'(u''_{\pi A}) \downarrow & & i'(u'_{pA}) \downarrow \\ j'R_2''(\pi A) & \xrightarrow{[R_2(A)]} & i'R_2(pA)[1] \end{array}$$

commutes in \mathcal{T} .

- (3) Suppose that Condition (i) of Proposition E.6.1 holds for \mathcal{B} . Then u exists and is unique if and only if the Condition in (1) is satisfied for $i = 1$.
- (4) u is a natural isomorphism if and only if u' and u'' are.

Proof. (1) This is clear for $n = 0$, and for $n = 1$ it follows by applying u to exact sequences of type (E.5).

(2) and (3) follow from Lemma E.1.1.

(4) “Only if” is obvious since u' and u'' are restrictions of u to \mathcal{A}' and \mathcal{A}'' , and “if” is obvious by the snake lemma. \square

E.11. The case of filtered categories. We now generalise Theorem E.10.2 to abelian categories provided with weight filtrations.

Let \mathcal{A}, \mathcal{B} be two abelian categories, filtered in the sense of Definition E.7.1. We assume that the filtrations are *weight filtrations* (Definition E.7.2).

We slightly change notation and consider two exact functors $\underline{\mathbb{T}}, \underline{\mathbb{T}}' : \mathcal{A} \rightarrow \mathcal{B}$ which respect the splittings in the sense of Definition E.9.1. By Lemma E.9.2 (2), we then have for every object $A \in \mathcal{A}$:

$$\underline{\mathbb{T}}(A_{\leq n}) = \underline{\mathbb{T}}(A)_{\leq n}, \quad \underline{\mathbb{T}}'(A_{\leq n}) = \underline{\mathbb{T}}'(A)_{\leq n}.$$

If $u : \underline{\mathbb{T}}' \Rightarrow \underline{\mathbb{T}}$ is a natural transformation, then u_A maps $\underline{\mathbb{T}}'(A_{\leq n})$ to $\underline{\mathbb{T}}(A_{\leq n})$, hence u induces natural transformations $u_{\leq n} : \underline{\mathbb{T}}'_{\leq n} \Rightarrow \underline{\mathbb{T}}_{\leq n}$ and $u_n : \underline{\mathbb{T}}'_n \Rightarrow \underline{\mathbb{T}}_n$.

E.11.1. Definition. For each $n \in \mathbb{Z}$, let $u_n : \underline{\mathbb{T}}'_n \Rightarrow \underline{\mathbb{T}}_n$ be a natural transformation. A *glueing* of the u_n is a natural transformation $u : \underline{\mathbb{T}}' \Rightarrow \underline{\mathbb{T}}$ which induces the u_n .

E.11.2. Theorem. (1) *There exists at most one glueing.*

- (2) Suppose that the glueing u exists on $\mathcal{A}_{\leq n-1}$. Then u extends to $\mathcal{A}_{\leq n}$ if and only if, for any $(A, B) \in \mathcal{A}_{\leq n-1} \times \mathcal{A}_n$, the diagram

$$\begin{array}{ccc} \mathrm{Ext}_{\mathcal{A}}^1(B, A) & \xrightarrow{\underline{T}'} & \mathrm{Ext}_{\mathcal{B}}^1(\underline{T}'(B), \underline{T}'(A)) \\ \underline{T} \downarrow & & u(A)_* \downarrow \\ \mathrm{Ext}_{\mathcal{B}}^1(\underline{T}(B), \underline{T}(A)) & \xrightarrow{u_n(B)^*} & \mathrm{Ext}_{\mathcal{B}}^1(\underline{T}'(B), \underline{T}(A)) \end{array}$$

commutes.

- (3) u is a natural isomorphism on objects of finite lengths if and only if all the u_n are.

Proof. (1) follows from Lemma E.7.7, while (2) follows from Theorem E.14.1 (2).

(3) “Only if” is obvious, and “if” follows inductively from Theorem E.14.1 (3). \square

E.12. The case of a 3-step filtration. We now restrict to the case where the filtration on \mathcal{A} has only 3 steps, and will try and get a condition involving only the u_n . For convenience, we assume that $\mathcal{A}_n \neq 0 \Rightarrow n \in \{-2, -1, 0\}$.

E.12.1. Theorem.

- (1) Suppose that, for $(m, n) \in \{(-2, -1), (-1, 0)\}$ and any $(A_m, A_n) \in \mathcal{A}_m \times \mathcal{A}_n$, the diagram

$$\begin{array}{ccc} \mathrm{Ext}_{\mathcal{A}}^1(A_n, A_m) & \xrightarrow{\underline{T}'} & \mathrm{Ext}_{\mathcal{B}}^1(\underline{T}'(A_n), \underline{T}'(A_m)) \\ \underline{T} \downarrow & & u(A_m)_* \downarrow \\ \mathrm{Ext}_{\mathcal{B}}^1(\underline{T}(A_n), \underline{T}(A_m)) & \xrightarrow{u_n(A_n)^*} & \mathrm{Ext}_{\mathcal{B}}^1(\underline{T}'(A_n), \underline{T}(A_m)) \end{array}$$

commutes or, equivalently, the obstruction in Theorem E.10.2 (1) vanishes. Let $u_{\leq -1}$ be the resulting natural transformation on $\mathcal{A}_{\leq -1}$ (Theorem E.10.2 (3)). Then, for any $(A_{\leq -1}, A_0) \in \mathcal{A}_{\leq -1} \times \mathcal{A}_0$, The glueing obstruction

$$\mathrm{Ext}_{\mathcal{A}}^1(A_0, A_{\leq -1}) \rightarrow \mathrm{Ext}_{\mathcal{B}}^1(\underline{T}'A_0, \underline{T}A_{\leq -1})$$

refines to an obstruction

$$\mathrm{Ext}_{\mathcal{A}}^1(A_0, A_{\leq -1}) \rightarrow \mathrm{Ext}_{\mathcal{B}}^1(\underline{T}'A_0, \underline{T}A_{-2}).$$

- (2) Suppose moreover that the diagram of (1) commutes for $(m, n) = (-2, 0)$ and that the map $\mathrm{Ext}_{\mathcal{A}}^1(A_0, A_{\leq -1}) \rightarrow \mathrm{Ext}_{\mathcal{A}}^1(A_0, A_{-1})$ is surjective for any $A_0 \in \mathcal{A}_0$ and $A_{\leq -1} \in \mathcal{A}_{\leq -1}$ (for example, that

\mathcal{A} is of cohomological dimension ≤ 1). Then the obstruction of (1) refines to a bilinear obstruction

$$\mathrm{Ext}_{\mathcal{A}}^1(A_0, A_{-1}) \times \mathrm{Ext}_{\mathcal{A}}^1(A_{-1}, A_{-2}) \rightarrow \mathrm{Ext}_{\mathcal{B}}^1(\underline{\mathbb{T}}'A_0, \underline{\mathbb{T}}A_{-2})$$

which is covariant in A_2 , contravariant in A_0 and “dinatural” in A_{-1} in the sense that the corresponding map

$$\mathrm{Ext}_{\mathcal{A}}^1(A_{-1}, A_{-2}) \rightarrow \mathrm{Hom}(\mathrm{Ext}_{\mathcal{A}}^1(A_0, A_{-1}), \mathrm{Ext}_{\mathcal{B}}^1(\underline{\mathbb{T}}'A_0, \underline{\mathbb{T}}A_{-2}))$$

is contravariant in A_{-1} .

- (3) If u_{-2} is a natural isomorphism, the obstruction of (2) may be reformulated as an obstruction with values in $\mathrm{Ext}_{\mathcal{B}}^1(\underline{\mathbb{T}}'A_0, \underline{\mathbb{T}}'A_{-2})$.
- (4) The glueing exists if and only if the obstruction of (2) vanishes.

Proof. (1) follows from Proposition E.3.3 (4). For (2), we have a commutative diagram

$$\begin{array}{ccc} \mathrm{Ext}_{\mathcal{A}}^1(A_0, A_{-2}) & \longrightarrow & \mathrm{Ext}_{\mathcal{B}}^1(\underline{\mathbb{T}}'A_0, \underline{\mathbb{T}}A_{-2}) \\ \downarrow & & \parallel \downarrow \\ \mathrm{Ext}_{\mathcal{A}}^1(A_0, A_{\leq -1}) & \longrightarrow & \mathrm{Ext}_{\mathcal{B}}^1(\underline{\mathbb{T}}'A_0, \underline{\mathbb{T}}A_{-2}) \end{array}$$

and the top horizontal map is 0 by hypothesis. Hence (by the surjectivity assumption) it induces a map $\mathrm{Ext}_{\mathcal{A}}^1(A_0, A_{-1}) \rightarrow \mathrm{Ext}_{\mathcal{B}}^1(\underline{\mathbb{T}}'A_0, \underline{\mathbb{T}}A_{-2})$, which is covariant in the second variable viewed as a functor of $A_{\leq -1}$. The result then follows from Lemma E.5.3.

(3) is obvious and (4) follows from Theorem E.11.2. \square

IV. GLUEING EQUIVALENCES OF ABELIAN CATEGORIES

E.13. The case of a 2-step filtration.

E.13.1. Theorem. Let $0 \rightarrow \mathcal{A}' \xrightarrow{i} \mathcal{A} \xrightarrow{\pi} \mathcal{A}'' \rightarrow 0$ and $0 \rightarrow \mathcal{B}' \xrightarrow{i'} \mathcal{B} \xrightarrow{\pi'} \mathcal{B}'' \rightarrow 0$ be as in (E.2), and consider a naturally commutative diagram of exact functors

$$\begin{array}{ccccccc} 0 & \longrightarrow & \mathcal{A}' & \xrightarrow{i} & \mathcal{A} & \xrightarrow{\pi} & \mathcal{A}'' \longrightarrow 0 \\ & & R' \downarrow & & R \downarrow & & R'' \downarrow \\ 0 & \longrightarrow & \mathcal{B}' & \xrightarrow{i'} & \mathcal{B} & \xrightarrow{\pi'} & \mathcal{B}'' \longrightarrow 0. \end{array}$$

Then:

- (1) If R' and R'' are faithful, R is faithful.
- (2) Assume the conditions of Proposition E.8.1 are satisfied and suppose further that, for any two objects $A' \in \mathcal{A}'$, $A'' \in \mathcal{A}''$, the map

$$\mathrm{Hom}_{\mathcal{A}}(jA'', iA') \rightarrow \mathrm{Hom}_{\mathcal{B}}(R(jA''), R(iA'))$$

is surjective and the map

$$(E.9) \quad \text{Ext}_{\mathcal{A}}^1(jA'', iA') \rightarrow \text{Ext}_{\mathcal{B}}^1(R(jA''), R(iA'))$$

is injective. Then, if R' and R'' are full, R is full.

- (3) Assume the conditions of (2) and suppose further that, for any two objects $A' \in \mathcal{A}'$, $A'' \in \mathcal{A}''$, the map (E.9) is surjective. Then, if R' and R'' are essentially surjective, R is essentially surjective.

Proof. (1) Let $f : A_1 \rightarrow A_2$ in \mathcal{A} be such that $R(f) = 0$. Since $R''\pi \simeq \pi'R$, $R''\pi(f) = 0$, hence $\pi(f) = 0$. By calculus of fractions, this implies that one can find $s : A_2 \rightarrow B$ with $\text{Ker}(s), \text{Coker}(s) \in i(\mathcal{A}')$ such that $sf = 0$. This means that f factors as a composition

$$A_1 \xrightarrow{g} iK \xrightarrow{h} A_2$$

with $iK = \text{Ker}(s)$ and h is a mono. Then $R(h)R(g) = 0$. Since R is exact, $R(h)$ is a mono, hence $R(h) = 0$ which implies that $R(iK) = i'R'(K) = 0$, hence $K = 0$ since R' and i' are faithful.

(2) Let $A_1, A_2 \in \mathcal{A}$ and let $g \in \mathcal{B}(R(A_1), R(A_2))$. By the functoriality of (E.5), we get a commutative diagram

$$\begin{array}{ccccccc} 0 & \longrightarrow & i'p'R(A_1) & \longrightarrow & R(A_1) & \longrightarrow & j'\pi'R(A_1) \longrightarrow 0 \\ & & \downarrow i'p'(g) & & \downarrow g & & \downarrow j'\pi'(g) \\ 0 & \longrightarrow & i'p'R(A_2) & \longrightarrow & R(A_2) & \longrightarrow & j'\pi'R(A_2) \longrightarrow 0. \end{array}$$

Using the equivalent conditions of (2), $i'p'(g)$ and $j'\pi'(g)$ respectively give maps

$$g' : i'R'(pA_1) \rightarrow i'R'(pA_2), \quad g'' : j'R''(\pi A_1) \rightarrow j'R''(\pi A_2).$$

By fullness, g' and g'' are induced by maps

$$f' : pA_1 \rightarrow pA_2, \quad f'' : \pi A_1 \rightarrow \pi A_2.$$

Now consider the diagram

$$\begin{array}{ccccccc} 0 & \longrightarrow & ipA_1 & \longrightarrow & A_1 & \longrightarrow & j\pi A_1 \longrightarrow 0 \\ & & \downarrow i(f') & & & & \downarrow j(f'') \\ 0 & \longrightarrow & ipA_2 & \longrightarrow & A_2 & \longrightarrow & j\pi A_2 \longrightarrow 0 \end{array}$$

Consider the extension classes $[A_r] \in \text{Ext}_{\mathcal{A}}^1(j\pi A_r, ipA_r)$ ($r = 1, 2$). Then a map f filling in this diagram exists if and only if

$$i(f')_*[A_1] = j(f'')^*[A_2] \in \text{Ext}_{\mathcal{A}}^1(j\pi A_1, ipA_2).$$

Hence, by the existence of g , this equality is true after applying the functor R , and therefore it holds by the injectivity assumption.

Now the map $R(f) - g$ induces a map $h : Rj\pi(A_1) \rightarrow Rip(A_2)$. By the surjectivity assumption, h is of the form $R(h')$, and then $R(f - h') = g$.

(3) Let $B \in \mathcal{B}$. Then $p'B$ is in the essential image of R'' and $\pi'B$ is in the essential image of R' . Using the exact sequence (E.5), Property (ii) of (2) and the surjectivity assumption, we get $B \simeq R(A)$ for some $A \in \mathcal{A}$. \square

Here is a converse to Theorem E.13.1:

E.13.2. Theorem. *Keep the notation of Theorem E.13.1. Then:*

- (1) *If R is faithful (resp. full), so is R' .*
- (2) *If R is faithful (resp. full) and the equivalent conditions of Proposition E.8.1 hold, so is R'' .*
- (3) *If R is essentially surjective, R'' is essentially surjective, and so is R' provided R'' is faithful or conservative.*

Proof. (1) and (2) are obvious, considering commutative squares of the type

$$\begin{array}{ccc} \mathcal{A}'(A'_1, A'_2) & \xrightarrow{\sim} & \mathcal{A}(iA'_1, iA'_2) \\ R' \downarrow & & \downarrow R \\ \mathcal{A}'(R'A'_1, R'A'_2) & \xrightarrow{\sim} & \mathcal{A}(RiA'_1, RiA'_2) \end{array}$$

for R' , and similarly for R'' . The first part of (3) is obvious. For the second one, let $B' \in \mathcal{B}'$. Write $i'B' \simeq RA$ for some $A \in \mathcal{A}$. Then $R''\pi A \simeq \pi'RA = 0$. The hypothesis implies $\pi A = 0$, hence $A \simeq iA'$ for some $A' \in \mathcal{A}'$. Now $i'B' \simeq RiA' \simeq i'R'A'$, hence $B' \simeq R'A'$. \square

E.13.3. Corollary. *With the notation of Theorem E.13.1, suppose that*

- (i) *R' and R'' are equivalences of categories;*
- (ii) *the conditions of Proposition E.8.1 are verified;*
- (iii) *for any two objects $A' \in \mathcal{A}'$, $A'' \in \mathcal{A}''$, the map*

$$\mathrm{Ext}_{\mathcal{A}}^i(jA'', iA') \rightarrow \mathrm{Ext}_{\mathcal{B}}^i(R(jA''), R(iA'))$$

is an isomorphism for $i = 0, 1$.

Then R is an equivalence of categories.

Conversely, if R is an equivalence of categories and the conditions of Proposition E.8.1 are verified, R' and R'' are equivalences of categories. \square

E.14. The general case. Here is now a version of Theorem E.13.1 for filtered abelian categories.

E.14.1. Theorem. *Let \mathcal{A} and \mathcal{B} be two abelian categories provided with exhaustive separated split filtrations $(\mathcal{A}_{\leq n}, \iota_n)$ and $(\mathcal{B}_{\leq n}, \iota'_n)$. Let $R : \mathcal{A} \rightarrow \mathcal{B}$ be an exact functor and assume that R respects the splittings in the sense of Definition E.9.1.*

- (1) *If R_n is faithful for every n , then R is faithful.*
- (2) *Assume that for any $m < n$ and any two objects $A_m \in \mathcal{A}_m$, $A_n \in \mathcal{A}_n$, the map*

$$\mathrm{Hom}_{\mathcal{A}}(A_n, A_m) \rightarrow \mathrm{Hom}_{\mathcal{B}}(R(A_n), R(A_m))$$

is surjective and the map

$$\mathrm{Ext}_{\mathcal{A}}^1(A_n, A_m) \rightarrow \mathrm{Ext}_{\mathcal{B}}^1(R(A_n), R(A_m))$$

is injective. Assume also that every object of \mathcal{A} is of finite length in the sense of Definition E.7.6. Then, if R_n is full for every n , R is full.

- (3) *Assume that, for any $n \in \mathbb{Z}$, the map*

$$\mathrm{Ext}_{\mathcal{A}}^1(A, B) \rightarrow \mathrm{Ext}_{\mathcal{B}}^1(R(A), R(B))$$

is surjective for $A \in \mathcal{A}_n$ and $B \in \mathcal{A}_{\leq n-1}$. Assume also that every object of \mathcal{B} is of finite length. Then, if R_n are essentially surjective for every n , R is essentially surjective.

- (4) *If $\mathrm{Ext}_{\mathcal{B}}^2(A, B) = 0$ for all $A, B \in \mathcal{B}$, we may weaken Condition (3) by requesting surjectivity for all $(A, B) \in \mathcal{A}_n \times \mathcal{A}_m$, $m < n$.*

Proof. (1) Let $f : A \rightarrow A'$ be such that $R(f) = 0$. There exists n_0 such that $A, A' \in \mathcal{A}_{n_0}$. By induction, Theorem E.13.1 implies that $f_{\geq n} = 0$ for all n (see Proposition E.7.4 (2) for $f_{\geq n}$). Thus, $\mathrm{Im} f \in \bigcap \mathcal{A}_{\leq n} = 0$.

(2) Since the filtrations are exhaustive, it suffices to prove that $R_{\leq n}$ is full for all n . Thus we may assume that $\mathcal{A}_{\leq n} = \mathcal{A}$ for n large enough. Similarly, since every object is of finite length, it suffices to prove that the restriction of R to the thick subcategory consisting of objects of length $\leq \ell$ is full for all $\ell \geq 0$. Thus, we may also assume that $\mathcal{A}_{\leq n'} = 0$ for n' small enough. By induction on $n - n'$, this reduces us to showing that the assumption of Theorem E.13.1 (3) are verified for $\mathcal{A}' = \mathcal{A}_{\leq n-1}$.

Let $A' \in \mathcal{A}_{\leq n-1}$ and $A'' \in \mathcal{A}_n$. We want to show that the map

$$\mathrm{Hom}_{\mathcal{A}}(A'', A') \rightarrow \mathrm{Hom}_{\mathcal{B}}(R(A''), R(A'))$$

is surjective and the map

$$\mathrm{Ext}_{\mathcal{A}}^1(A'', A') \rightarrow \mathrm{Ext}_{\mathcal{B}}^1(R(A''), R(A'))$$

is injective. Consider the short exact sequence $0 \rightarrow A'_{\leq n-2} \rightarrow A' \rightarrow A'_{n-1} \rightarrow 0$. The statement is true by replacing A' by $A'_{\leq n-2}$ (by induction) or by A'_{n-1} (by hypothesis). Hence it is true for A' , by the five lemma.

(3) We argue as in (2): since the filtration of \mathcal{B} is exhaustive, it suffices to prove that $R_{\leq n}$ is essentially surjective for all n , hence we may assume $\mathcal{B}_{\leq n} = \mathcal{B}$ for n large enough. Since every object of \mathcal{B} has finite length, it suffices to prove that every object of \mathcal{B} of length $\leq \ell$ is isomorphic to $R(A)$, where $A \in \mathcal{A}$ is of length $\leq \ell$. Thus we may also assume that $\mathcal{A}_{\leq n'} = \mathcal{B}_{\leq n'} = 0$ for n' small enough. We argue by induction on $n - n'$: by induction, $R_{\leq n-1}$ is essentially surjective, and so is R_n by hypothesis. On the other hand, the assumptions of Theorem E.13.1 (4) are verified for $\mathcal{A}' = \mathcal{A}_{\leq n-1}$ and $\mathcal{B}' = \mathcal{B}_{\leq n-1}$.

(4) By the 5 lemma, we deduce the condition of (3) from this weaker condition. \square

V. THE CASE OF TRIANGULATED CATEGORIES

E.15. Split exact sequences. We let \mathcal{T} be a triangulated category. Consider the situation

$$(E.10) \quad 0 \rightarrow \mathcal{T}' \xrightarrow{i} \mathcal{T} \xrightarrow{\pi} \mathcal{T}'' \rightarrow 0$$

where $\mathcal{T}' \subseteq \mathcal{T}$ is a thick subcategory and π is the corresponding localisation functor. Thus, \mathcal{T}'' is the Verdier quotient of \mathcal{T} by \mathcal{T}' .

E.15.1. Proposition (Verdier). *The following conditions are equivalent:*

- (i) *i has a right adjoint p .*
- (ii) *π has a right adjoint j and $j(\mathcal{T}'')$ is thick in \mathcal{T} .*
- (iii) *for any $A \in \mathcal{T}$ there exists $A' \in \mathcal{T}'$ and a map $f : i(A') \rightarrow A$ such that $\text{Hom}(i(\mathcal{T}'), \text{cone}(f)) = 0$.*

If these conditions are verified, then for any $A \in \mathcal{T}$ there exists a unique exact triangle

$$A' \rightarrow A \rightarrow A'' \xrightarrow{+1}$$

such that $A' \in i(\mathcal{T}')$ and $A'' \in j(\mathcal{T}'')$; this triangle is functorial in A . Moreover, we have $\text{Hom}_{\mathcal{T}}(i(\mathcal{T}'), j(\mathcal{T}'')) = 0$.

Proof. (i) \Rightarrow (ii): let $A \in \mathcal{T}$, and choose a cone C_A of the counit $\varepsilon_A : ipA \rightarrow A$. Applying p to the exact triangle

$$ipA \xrightarrow{\varepsilon_A} A \rightarrow C_A \xrightarrow{+1}$$

we get an exact triangle

$$pA \xrightarrow{=} pA \rightarrow pC_A \xrightarrow{+1}$$

which shows that

$$(E.11) \quad pC_A = 0.$$

Let $f : A \rightarrow B$ be a morphism, and choose a cone C_B analogously. There exists a morphism g such that the diagram of exact triangles

$$\begin{array}{ccccc} ipA & \xrightarrow{\varepsilon_A} & A & \longrightarrow & C_A \xrightarrow{+1} \\ ipf \downarrow & & f \downarrow & & g \downarrow \\ ipB & \xrightarrow{\varepsilon_B} & B & \longrightarrow & C_B \xrightarrow{+1} \end{array}$$

commutes.

Applying (E.11) to C_B , we get by adjunction $\mathrm{Hom}_{\mathcal{T}}(ipA[1], C_B) = 0$, which shows that g is *unique*; in particular, taking $f = 1_A$, we get that C_A is defined up to unique isomorphism.

Letting $\mathcal{S} := \mathrm{Ker} p$, this thus defines a functor $q : \mathcal{T} \rightarrow \mathcal{S}$ ($qA = C_A$), and q is easily seen to be left adjoint to the inclusion $\mathcal{S} \hookrightarrow \mathcal{T}$. Moreover $q|_{\mathcal{T}'} = 0$, hence q induces a functor $\bar{q} : \mathcal{T}'' \rightarrow \mathcal{S}$.

On the other hand, we have the obvious functor

$$r : \mathcal{S} \rightarrow \mathcal{T} \rightarrow \mathcal{T}''.$$

Let $B \in \mathcal{S}$. By definition, $\bar{q}rB$ is represented by $C_B = B$. This provides a natural isomorphism $B \xrightarrow{\sim} \bar{q}rB$.

Let $A \in \mathcal{T}$. The map $A \rightarrow C_A$ induces another natural isomorphism $A \xrightarrow{\sim} r\bar{q}A$, where A is now viewed in \mathcal{T}'' . So \bar{q} and r are quasi-inverse equivalences of categories. Under this equivalence, q becomes π , and the inclusion $\mathcal{S} \hookrightarrow \mathcal{T}$ defines a functor $j : \mathcal{T}'' \rightarrow \mathcal{T}$, which is right adjoint to π . Finally, since $\mathcal{S} = \mathrm{Ker} p$ is thick, $j(\mathcal{T}'')$ is thick.

(ii) \implies (iii): let $A \in \mathcal{T}$. The functor π kills the cone D_A of the unit map $\eta_A : A \rightarrow j\pi A$. Hence $D_A \in i(\mathcal{T}')$. If $A'_1 \in \mathcal{T}'$, then

$$\mathrm{Hom}(iA'_1, j\pi A) = \mathrm{Hom}(\pi iA'_1, \pi A) = \mathrm{Hom}(0, \pi A) = 0$$

hence we may take $A' = D_A[-1]$ in (iii).

(iii) \implies (i): one checks that $A \mapsto A'$ yields the desired adjoint, by the same kind of arguments as in the proof of (i) \implies (ii).

Finally, the claimed exact triangle is given by

$$(E.12) \quad ipA \xrightarrow{\varepsilon_A} A \xrightarrow{\eta_A} j\pi A \xrightarrow{\delta} ipA[1]$$

and its properties follow by construction. The vanishing of $\mathrm{Hom}_{\mathcal{T}}(i\mathcal{T}', j\mathcal{T}'')$ follows from the adjunction. \square

E.15.2. Definition. In the situation of Proposition E.15.1, we say that the exact sequence (E.10) is *split*.

E.16. More adjoints. Assume that (E.10) is split. Let us now consider the conditions

- (b) j has a right adjoint π' .
- (#) i has a left adjoint p' .

Recall that, under (b), we get a canonical natural transformation $\alpha : \pi' \Rightarrow \pi$ from the composition $j\pi'A \rightarrow A \rightarrow j\pi A$ for any $A \in \mathcal{T}$, and the full faithfulness of j . Similarly, under (#), we get a canonical natural transformation $\beta : p \Rightarrow p'$.

E.16.1. Proposition. *Consider the following conditions:*

- (i) $\text{Hom}_{\mathcal{T}}(j\mathcal{T}'', i\mathcal{T}') = 0$.
- (ii) (b) holds and $\pi' = \pi$.
- (iii) (b) holds and $\pi'i = 0$.
- (iv) (#) holds and $p' = p$.
- (v) (#) holds and $p'j = 0$.

Then (i) + (b) \iff (i) + (#) \iff (ii) \iff (iii) \iff (iv) \iff (v).

Proof. Suppose that π' exists. Let $A \in \mathcal{T}$. Applying π' to the exact triangle (E.12), we get an exact triangle

$$\pi'ipA \rightarrow \pi'A \xrightarrow{\alpha_A} \pi A \xrightarrow{+1}$$

This shows that α_A is an isomorphism if and only if $\pi'ipA = 0$. Thus, $(\alpha_A \text{ injective for all } A) \iff (\pi'ipA = 0 \text{ for all } A) \iff (\pi'i = 0)$ since p is essentially surjective. Thus, (ii) \iff (iii). On the other hand, (i) + (b) \iff (iii) is obvious by adjunction. The other equivalences are obtained dually. \square

E.17. Filtered triangulated categories.

E.17.1. Definition. Let \mathcal{T} be an triangulated category.

a) A *filtration* on \mathcal{T} is a sequence of thick subcategories

$$\dots \xrightarrow{\iota_{n-1}} \mathcal{T}_{\leq n} \xrightarrow{\iota_n} \mathcal{T}_{\leq n+1} \xrightarrow{\iota_{n+1}} \dots \rightarrow \mathcal{T}.$$

It is *separated* if $\bigcap \mathcal{T}_{\leq n} = \{0\}$ and *exhaustive* if $\mathcal{T} = \bigcup \mathcal{T}_{\leq n}$.

b) A filtration on \mathcal{T} is *split* if all ι_n have exact right adjoints.

c) A filtration is a *weight filtration* if it is separated, exhaustive and split.

Let \mathcal{T} be provided with a filtration. Define

$$\mathcal{T}_n = \mathcal{T}_{\leq n} / \mathcal{T}_{\leq n-1}, \quad \mathcal{T}_{n,n+1} = \mathcal{T}_{\leq n+1} / \mathcal{T}_{\leq n-1}$$

so that we have exact sequences

$$0 \rightarrow \mathcal{T}_n \xrightarrow{i_n} \mathcal{T}_{n,n+1} \xrightarrow{\pi_n} \mathcal{T}_{n+1} \rightarrow 0$$

where i_n is induced by ι_n .

E.17.2. Proposition. *Suppose that \mathcal{T} is provided with an exhaustive split filtration as in Definition E.17.1 a), b). For all $n \in \mathbb{Z}$, let us write j_n for the right adjoint of the localisation functor $\mathcal{T}_{\leq n} \rightarrow \mathcal{T}_{\leq n}/\mathcal{T}_{\leq n-1}$ (Proposition E.15.1). By abuse of notation, we shall write \mathcal{T}_n for the thick subcategory $j_n(\mathcal{T}_{\leq n}/\mathcal{T}_{\leq n-1}) \subset \mathcal{T}$ (ibid.). Then:*

- (1) *Every object $A \in \mathcal{T}$ is provided with a unique filtration*

$$\cdots \rightarrow A_{\leq n} \rightarrow A_{\leq n+1} \rightarrow \cdots \rightarrow A$$

with $A_{\leq n} \in \mathcal{T}_{\leq n}$ and $A_n := \text{cone}(A_{\leq n-1} \rightarrow A_{\leq n}) \in \mathcal{T}_n$. One has $A_{\leq n} = A$ for n large enough.

- (2) *The functors $A \mapsto A_{\leq n}$ and $A \mapsto A_n$ are well-defined, as well as $A \mapsto A_{\geq n} := A/A_{\leq n-1}$.*
 (3) *If moreover the filtration is separated, the $A \mapsto A_n$ form a set of conservative functors.*
 (4) *We have $\text{Hom}_{\mathcal{T}}(\mathcal{T}_m, \mathcal{T}_n) = 0$ if $m < n$.*

Proof. (1) Since $\mathcal{T} = \bigcup \mathcal{T}_{\leq n}$, there exists n_0 such that $A \in \mathcal{T}_{\leq n_0}$. For $n < n_0$, write I_n for the inclusion $\mathcal{T}_{\leq n} \rightarrow \mathcal{T}_{\leq n_0}$ and P_n for its right adjoint, which exists and is exact as a composition of the right adjoints of the ι_r for $n \leq r < n_0$. Define

$$A_{\leq n} = I_n P_n A.$$

Since $A = \iota_{n_0} \varpi_{n_0} A$, $A_{\leq n}$ does not depend on the choice of n_0 , and we have a filtration of A as requested. Clearly, $A_{\leq n} \in \mathcal{T}_{\leq n}$ and $A_{\leq n_0} = A$. The fact that $A_n \in \mathcal{T}_n$ follows from Proposition E.15.1, which also shows the uniqueness of the filtration.

(2) This still follows from Proposition E.15.1.

(3) Let $f : A \rightarrow B$ be such that $f_n : A_n \rightarrow B_n$ is an isomorphism for all $n \in \mathbb{Z}$. Let $C = \text{cone}(f)$. Then $C_n = 0$ for all n . Thus $C \in \bigcap \mathcal{T}_{\leq n} = 0$ and f is an isomorphism.

(4) It suffices to show that $\text{Hom}_{\mathcal{T}}(\mathcal{T}_{\leq n-1}, \mathcal{T}_n) = 0$, which follows again from Proposition E.15.1. \square

E.18. Glueing equivalences of triangulated categories.

E.18.1. Theorem. *Let $0 \rightarrow \mathcal{S}' \xrightarrow{i} \mathcal{S} \xrightarrow{\pi} \mathcal{S}'' \rightarrow 0$ and $0 \rightarrow \mathcal{T}' \xrightarrow{i'} \mathcal{T} \xrightarrow{\pi'} \mathcal{T}'' \rightarrow 0$ be as in (E.10), and consider a naturally commutative diagram of exact functors*

$$\begin{array}{ccccccc} 0 & \longrightarrow & \mathcal{S}' & \xrightarrow{i} & \mathcal{S} & \xrightarrow{\pi} & \mathcal{S}'' \longrightarrow 0 \\ & & R' \downarrow & & R \downarrow & & R'' \downarrow \\ 0 & \longrightarrow & \mathcal{T}' & \xrightarrow{i'} & \mathcal{T} & \xrightarrow{\pi'} & \mathcal{T}'' \longrightarrow 0. \end{array}$$

- (1) *The following are equivalent:*
- (i) *The natural “base change” transformation $R'p \Rightarrow p'R$ of Section 15.1 is a natural isomorphism.*
 - (ii) *$Rj \Rightarrow j'R''$ is a natural isomorphism.*
 - (iii) *$R(j\mathcal{S}'') \subseteq j'\mathcal{T}''$.*
- (2) *Suppose that*
- (i) *R' and R'' are fully faithful;*
 - (ii) *the conditions of (1) are verified;*
 - (iii) *for any two objects $A' \in \mathcal{S}'$, $A'' \in \mathcal{S}''$, the map*

$$\mathrm{Hom}_{\mathcal{S}}(jA'', iA') \rightarrow \mathrm{Hom}_{\mathcal{T}}(R(jA''), R(iA'))$$

is an isomorphism.

Then R is fully faithful.

- (3) *Suppose that Condition (iii) of (2) holds and that R' and R'' are essentially surjective. Then so is R .*
- (4) *Suppose that R' and R'' are equivalences of categories and that the conditions of (1) and Condition (iii) of (2) are satisfied. Then R is an equivalence of categories.*

Proof. (1) Let $A \in \mathcal{S}$. We have a commutative diagram of exact triangles (using (E.12) and the base change morphisms):

$$\begin{array}{ccccc} R(ipA) & \longrightarrow & R(A) & \longrightarrow & R(j\pi A) \xrightarrow{+1} \\ \downarrow & & \parallel \downarrow & & \downarrow \\ i'p'R(A) & \longrightarrow & R(A) & \longrightarrow & j'\pi'R(A) \xrightarrow{+1} \end{array} .$$

Thus the left vertical map is an isomorphism if and only if the right vertical map is one, which shows that (i) \iff (ii). If this is the case, then $R(j\pi A) \in j'(\mathcal{T}')$, hence (iii). Conversely, if (iii) holds, then all vertical maps must be isomorphisms by the uniqueness of (E.12).

(2) A. We first show the fullness of R . Let $A_1, A_2 \in \mathcal{S}$ and let $g \in \mathcal{T}(R(A_1), R(A_2))$. By the functoriality of (E.12), we get a commutative diagram

$$\begin{array}{ccccc} i'p'R(A_1) & \longrightarrow & R(A_1) & \longrightarrow & j'\pi'R(A_1) \xrightarrow{+1} \\ i'p'(g) \downarrow & & g \downarrow & & j'\pi'(g) \downarrow \\ i'p'R(A_2) & \longrightarrow & R(A_2) & \longrightarrow & j'\pi'R(A_2) \xrightarrow{+1} \end{array} .$$

Using the equivalent conditions of (1), $i'p'(g)$ and $j'\pi'(g)$ respectively give maps

$$g' : i'R'(pA_1) \rightarrow i'R'(pA_2), \quad g'' : j'R''(\pi A_1) \rightarrow j'R''(\pi A_2).$$

By the fullness of R' and R'' , g' and g'' are induced by maps

$$f' : pA_1 \rightarrow pA_2, \quad f'' : \pi A_1 \rightarrow \pi A_2.$$

Now consider the diagram

$$\begin{array}{ccccc} A_1 & \longrightarrow & j\pi A_1 & \longrightarrow & ipA_1[1] \xrightarrow{+1} \\ & & j(f'') \downarrow & & i(f'[1]) \downarrow \\ A_2 & \longrightarrow & j\pi A_2 & \longrightarrow & ipA_2[1] \xrightarrow{+1} \end{array}$$

By the injectivity in Condition (iii), the square commutes, hence there exists a map $f : A_1 \rightarrow A_2$ filling in this diagram. Now the map $R(f) - g$ induces a map $h : Rj\pi(A_1) \rightarrow Rip(A_2)$. By the surjectivity in Condition (iii), h is of the form $R(h')$, and then $R(f - h') = g$. Thus R is full.

B. To see that R is faithful, it now suffices by A.1.1 to show that it is conservative. Let $A \in \mathcal{S}$ be such that $RA = 0$. Then $R''\pi A = 0$, hence $\pi A = 0$ since R'' , being fully faithful, is conservative. Hence $A \simeq iA'$ for some $A' \in \mathcal{S}'$. Then, $RiA' \simeq iR'A' = 0$, hence $A' = 0$ by the conservativity of i and R' and $A = 0$.

(3) Let $B \in \mathcal{T}$. Then $p'B$ is in the essential image of R'' and $\pi'B$ is in the essential image of R' . Thus B fits in an exact triangle of the form

$$B \rightarrow R(jA'') \xrightarrow{\delta'} R(iA'[1]) \xrightarrow{\pm 1}.$$

By the surjectivity in (2) (iii), δ' is of the form $R(\delta)$ for $\delta : jA'' \rightarrow iA'[1]$. Let A denote a fibre (= shifted cone) of δ . We can then fill in the diagram

$$\begin{array}{ccccc} R(iA') & \longrightarrow & R(A) & \longrightarrow & R(jA'') \xrightarrow{R(\delta)} \\ \parallel \downarrow & & & & \parallel \downarrow \\ R(iA') & \longrightarrow & B & \longrightarrow & R(jA'') \xrightarrow{R(\delta)} \end{array}$$

with a map which is automatically an isomorphism.

(4) This is just collecting the previous results. □

Here is a converse to Theorem E.18.1:

E.18.2. Theorem. *Keep the notation of Theorem E.18.1. Then:*

- (1) *If R is faithful (resp. full), so is R' .*
- (2) *If R is faithful (resp. full) and the equivalent conditions of Theorem E.18.1 (1) hold, so is R'' .*
- (3) *If R is essentially surjective, R'' is essentially surjective, and so is R' provided R'' is conservative.*

Proof. Same as for Theorem E.13.2. \square

VI. THE CASE OF t -CATEGORIES

Let \mathcal{T} be a triangulated category provided with a t -structure with heart \mathcal{A} [10, §1].

E.19. t -exact functors.

E.19.1. Lemma. *Let \mathcal{S} be another t -category, with heart \mathcal{B} , and let $T : \mathcal{S} \rightarrow \mathcal{T}$ be a triangulated functor. Assume that $T(\mathcal{B}) \subseteq \mathcal{A}$ and that the t -structure on \mathcal{S} is bounded. Then:*

- (1) T is t -exact;
- (2) the induced functor $T : \mathcal{B} \rightarrow \mathcal{A}$ is exact.

Proof. (1) Let us show right exactness. Let $X \in \mathcal{S}_{\geq 0}$. We must show that $T(X) \in \mathcal{T}_{\geq 0}$. By assumption, there exists $n \geq 0$ such that $X \in \mathcal{S}_{[0, n]}$. For $n = 0$, we have $T(X) \in \mathcal{A}$ by hypothesis. For $n > 0$, we may argue by induction on n , using the exact triangle

$$H_n(X)[n] \rightarrow X \rightarrow Y \xrightarrow{+1}$$

with $Y \in \mathcal{S}_{[0, n-1]}$. Left exactness is proven similarly.

- (2) This follows from [10, Prop. 1.3.17 (i)]. \square

E.20. The case of a 2-step filtration. Here we assume that \mathcal{A} sits in a short exact sequence (E.2).

E.20.1. Proposition. *Let \mathcal{T}' be the full subcategory of \mathcal{T} consisting of objects T such that $H^i(T) \in \mathcal{A}'$ for all $i \in \mathbb{Z}$. Then:*

- (1) \mathcal{T}' is thick in \mathcal{T} .
- (2) The t -structure of \mathcal{T} induces a t -structure on \mathcal{T}' .
- (3) Suppose that (E.2) is split in the sense of Definition E.5.2, and that the t -structure of \mathcal{T} is bounded. Then
 - (i) The exact sequence

$$0 \rightarrow \mathcal{T}' \xrightarrow{i} \mathcal{T} \xrightarrow{\pi} \mathcal{T}/\mathcal{T}' \rightarrow 0$$

is split in the sense of Definition E.15.2, and the right adjoint p to i is t -exact.

- (ii) *Via the right adjoint j to π (see Proposition E.15.1), the t -structure of \mathcal{T} induces a t -structure on \mathcal{T}/\mathcal{T}' , for which π is t -exact. Moreover, $j(\mathcal{T}/\mathcal{T}') = \{C \in \mathcal{T} \mid H^*(C) \in j(\mathcal{A}'')\}$.*

Proof. (1) The thickness of \mathcal{T}' in \mathcal{T} follows from the thickness of \mathcal{A}' in \mathcal{A} . (2) is clear.

It remains to prove (3).

(i) To prove that i has a right adjoint p , it suffices to prove Condition (iii) of Proposition E.15.1 (compare [64, lemmes 3.2.1 and 3.2.3]).

Let \mathcal{S} be the full subcategory of \mathcal{T} consisting of those objects verifying Condition (iii) of Proposition E.15.1. We have to show that $\mathcal{S} = \mathcal{T}$. Since the t -structure is supposed to be bounded, it suffices to check that \mathcal{S} is triangulated and contains \mathcal{A} .

It is clear that \mathcal{S} is stable under shifts. Let $A_1, A_2 \in \mathcal{S}$, $f : A_1 \rightarrow A_2$ and $A'_1, A'_2 \in \mathcal{T}'$ satisfying Condition (iii) of Proposition E.15.1 respectively for A_1 and A_2 . This condition implies that the composition $A'_1 \rightarrow A_1 \rightarrow A_2$ factors through a (unique) map $f' : A'_1 \rightarrow A'_2$. We also get a map $f'' : \text{cone}(f_1) \rightarrow \text{cone}(f_2)$ defining a map of exact triangles. Let $A'_3 = \text{cone}(f')$: we may find a map $f_3 : A'_3 \rightarrow A_3$ defining a morphism of exact triangles. Then the cone of f_3 is isomorphic to the cone of f'' , hence has the property of Proposition E.15.1. This shows that \mathcal{S} is triangulated.

Let $A \in \mathcal{A}$. By definition of “split” and Proposition E.5.1 (2), A sits in a short exact sequence $0 \rightarrow A' \rightarrow A \rightarrow A'' \rightarrow 0$ with $A' \in \mathcal{A}'$ and $\text{Ext}_{\mathcal{A}}^r(B', A'') = 0$ for all $B' \in \mathcal{A}'$ and all $r \geq 0$. Since the t -structure is bounded, this implies that $\text{Hom}_{\mathcal{T}}(\mathcal{T}', A'') = 0$, and $A \in \mathcal{S}$.

This shows the existence of the right adjoint p . Moreover, the proof of Proposition E.15.1 shows that, for A, A', A'' as in the last paragraph, $p(A) = A'$. Thus p respects the hearts of the t -structures on \mathcal{T} and \mathcal{T}' , and since the one on \mathcal{T} is bounded, p is t -exact by Lemma E.19.1.

(ii) To show that the t -structure of \mathcal{T} induces a t -structure on $j(\mathcal{T}/\mathcal{T}')$, we have to show that, for $X \in j(\mathcal{T}/\mathcal{T}')$, $\tau_{\leq 0}X \in j(\mathcal{T}/\mathcal{T}')$, or equivalently that if $p(X) = 0$, then $p(\tau_{\leq 0}X) = 0$: this is clear since p is t -exact. The proof that π is t -exact is then the same as the proof of the t -exactness of p .

For the last assertion, let $C \in j(\mathcal{T}/\mathcal{T}')$. Then, for any $i \in \mathbb{Z}$, $H^i(j(C)) = j(H^i(C)) \in j(\mathcal{A}'')$. Conversely, if $C \in \mathcal{T}$ is such that $H^i(C) \in j(\mathcal{A}'')$ for all $i \in \mathbb{Z}$, then $pH^i(C) = H^i(pC) = 0$ for all $i \in \mathbb{Z}$, hence $pC = 0$ since the t -structure is bounded, and $C \in \text{Im } j$ by Proposition E.15.1. \square

E.21. Filtered t -categories. We now assume that \mathcal{A} is filtered in the sense of Definition E.7.1 a).

E.21.1. Definition. Let $n \in \mathbb{Z}$. We denote by $\mathcal{T}_{\leq n}$ the full subcategory of \mathcal{T} consisting of objects T such that $H^i(T) \in \mathcal{A}_{\leq n}$ for all $i \in \mathbb{Z}$.

This definition is similar to that of Sophie Morel in [64, Prop. 3.1.1].

E.21.2. Proposition.

- (1) The $\mathcal{T}_{\leq n}$ define a filtration of \mathcal{T} in the sense of Definition E.17.1 a).
- (2) For any n , the t -structure of \mathcal{T} induces a t -structure on $\mathcal{T}_{\leq n}$.
- (3) It is split if the filtration on \mathcal{A} is and the t -structure is bounded.
- (4) It is separated if the filtration on \mathcal{A} is, and if the t -structure is nondegenerate.
- (5) It is exhaustive if the filtration on \mathcal{A} is and if the t -structure is bounded.

Proof. (1), (2) and (3) follow respectively from Proposition E.20.1 (1), (2) and (3). (4) and (5) are easy. \square

E.21.3. Remark. As in [40, Def. 1.2.1], let \mathcal{T} be a t -category with compatible triangulated endofunctors W_n with commute with the t -structure and such that the transformations $W_n \rightarrow W_{n+1}$ and $W_n \rightarrow Id$ are inclusions on the heart \mathcal{A} of \mathcal{T} . In particular, the W_n induce an increasing sequence of *exact* subfunctors of the identity of \mathcal{A} . Thus we are in the situation of Remark E.7.8 if they are separated and exhaustive on all objects of \mathcal{A} . By this remark, we then have a weight filtration on \mathcal{A} in the sense of Definition E.7.2.

Define $\mathcal{T}_{\leq n}$ as in Definition E.21.1. By Proposition E.21.2, this is a weight filtration on \mathcal{T} in the sense of Definition E.17.1 c). By Proposition E.17.2, we get endofunctors of \mathcal{T} :

$$W'_n C = C_{\leq n}.$$

We claim that $W'_n C = W_n C$. First, $W_n C \in \mathcal{T}_{\leq n}$ because W_n commutes with t and this is true on \mathcal{A} by definition of $\mathcal{A}_{\leq n}$. By the universal property of $W'_n C$, the map $W_n C \rightarrow C$ factors uniquely into

$$W_n C \rightarrow W'_n C.$$

That this is an isomorphism is checked again after truncation, and it is true by Remark E.7.8.

Summarising: given a t -category (\mathcal{T}, t) , a weight filtration (W_n) in the sense of [40, Def. 1.2.1] is equivalent to a weight filtration on \mathcal{T} in the sense of Definition E.17.1 c) which is compatible with the t -structure, provided that the weight filtration is separated and exhaustive on objects of the heart.

E.22. From abelian categories to t -categories.

E.22.1. Lemma. *Let \mathcal{A}, \mathcal{B} be two abelian categories, $\underline{\mathbb{T}}, \underline{\mathbb{T}}' : \mathcal{A} \rightarrow \mathcal{B}$ be two exact functors and $u_1 : \underline{\mathbb{T}}' \Rightarrow \underline{\mathbb{T}}$ be a natural transformation. Then u extends uniquely to a natural transformation u between the trivial derivations of $\underline{\mathbb{T}}'$ and $\underline{\mathbb{T}}$ from $D^b(\mathcal{A})$ to $D^b(\mathcal{B})$. Moreover, u is an isomorphism if and only if u_1 is.*

Proof. Trivial (define u componentwise on chain complexes). \square

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